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Exercise 1.16

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Let $P(n)$ be the average number of subarrays of size zero that appear while partitioning an array of size n, $Q(n)$ be the average number for size 1, and $R(n)$ be for size 2.

Since each partition spot has an equal chance of being chosen, we have

$$P(n) = \frac{1}{n}\sum_{i=1}^{n}(P(i-1) + P(n-i))$$
$$P(n) = \frac{2}{n}\sum_{i=0}^{n-1}P(i)$$
$$nP(n) = 2\sum_{i=0}^{n-1}P(i)$$

$$nP(n) - (n-1)P(n-1) = 2P(n-1)$$

$$nP(n) = (n+1)P(n-1)$$

$$P(n) = \frac{n+1}{n}P(n-1)$$

This relation holds for all three functions. However, the initial conditions are different for each.

$P(0) = 1,\ P(1) = 0$, since there the only possible subarray has size 1. $P(2) = 1$ since $P(2) = \frac{2}{2}\sum_{i=0}^{1}P(i) = 1(1+0)$ by the original recurrence.

The recurrence relation ends up simplifying to $P(n) = \frac{n+1}{3}$

n	1	2	3	4
$P(n)$	0	1	$\frac{4}{3}$	$\frac{5}{3}$

$Q(0) = 0,\ Q(1) = 1$, logically, and $Q(2) = 1$ because we end up with the same computation as above, with $P(n)$.

The recurrence simplifies to $Q(n) = \frac{n+1}{3}$

n	1	2	3	4
$Q(n)$	1	1	$\frac{4}{3}$	$\frac{5}{3}$

$R(0) = R(1) = 0$ since subarrays of size two are not even possible, and $R(2) = 1$. By using the original formula, we have $R(3) = \frac{2}{3}(0 + 0 + 1) = \frac{2}{3}$, which gives us $R(n) = \frac{n+1}{6}$ after simplifying the reduced recurrence relation.

n	1	2	3	4
$R(n)$	0	1	$\frac{2}{3}$	$\frac{5}{6}$

-1 Final answer? Assuming you meant to give P(n) + Q(n) + R(n) = 5(n+1)/6?