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Analytic Combinatorics Homework 1 Problem 1

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14. Multiplying the equation by N and plugging in both $N = N$ and $N = N+1$, we have

$$(N+1)A_{N+1} = N + 1 + 2\sum_{1\le j\le N+1} A_{j-1}$$
$$NA_N = N + 2\sum_{1\le j\le N} A_{j-1}.$$

Subtracting, we get $(N+1)A_{N+1} - NA_N = 1 + 2A_N$, so $A_{N+1} = \frac{(N+2)A_N+1}{N+1}$. We show inductively that $A_N = N$. We have $A_0 = 0$. Suppose that $A_N = N$ for all $N \le n$. Then $A_{n+1} = \frac{(n+2)n+1}{n+1} = \frac{(n+1)^2}{n+1} = n+1$, completing our induction. Thus, we have $\boxed{A_N = N}$.

Let X_N be the average number of times that quicksort is called with $hi = lo$ for an array of size N. Observe that the pivot has probability $\frac{1}{N}$ of ending up in position j, at which point quicksort is called on arrays of size $j-1$ and $N-j$. Each of these arrays has its elements in a random order, independent of j. Thus we have the recurrence relation

$$X_N = \sum_{j=1}^{N} \frac{1}{N}(X_{j-1} + X_{N-j}) = \frac{2}{N}\sum_{j=1}^{N} X_{j-1}$$

for $N \ge 2$, with $X_0 = 0$ and $X_1 = 1$. (This recurrence relation is not right for $N = 1$ because of the initial call of quicksort.) Again multiplying by N, plugging in $N = N$ and $N = N+1$, and subtracting gives us $(N+1)X_{N+1} - NX_N = 2X_N$, i.e.

$$X_{N+1} = \frac{N+2}{N+1}X_N.$$

We claim that for $N \ge 2$ we have $X_N = \frac{N+1}{3}$. We show this inductively. We have $X_2 = \frac{2}{2}(X_0 + X_1) = 1 = \frac{2+1}{3}$. Suppose that for $2 \le N \le n$ we have $X_N = \frac{N+1}{3}$. Then $X_{n+1} = \frac{n+2}{n+1}\cdot\frac{n+1}{3} = \frac{n+2}{3}$, completing our induction. Therefore, we have

$$\boxed{X_0 = 0, X_1 = 1, X_N = \frac{N+1}{3} \quad \forall N \ge 2}.$$