Analytic Combinatorics Homework 1 Problem 2

Eric Neyman
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16. Let Y_N be the average number of subarrays of size 2 or less encountered, on average, when sorting an array of size N with quicksort. Observe that the pivot has probability $\frac{1}{N}$ of ending up in position j, at which point quicksort is called on arrays of size $j-1$ and $N-j$. Each of these arrays has its elements in a random order, independent of j. Thus we have the recurrence relation

$$Y_N = \sum_{j=1}^{N} \frac{1}{N}(Y_{j-1} + Y_{N-j}) = \frac{2}{N}\sum_{j=1}^{N} Y_{j-1}$$

for $N \geq 3$. (This recurrence relation is not right for $N \leq 2$ because of the initial call of quicksort.) Multiplying by N and plugging in $N = N$ and $N = N+1$, we have

$$(N+1)Y_{N+1} = 2\sum_{j=1}^{N+1} Y_{j-1}$$
$$NY_N = 2\sum_{j=1}^{N} Y_{j-1}.$$

Subtracting, we get that $(N+1)Y_{N+1} - NY_N = 2Y_N$, so $Y_{N+1} = \frac{N+2}{N+1}Y_N$ for $N \geq 3$. Clearly $Y_0 = 1$ (the initial call on an array of size zero and then termination); $Y_1 = 1$ (the initial call on an array of size one and then termination); and $Y_2 = 3$ (one array of size two; one array of size one, i.e. the array consisting of the non-pivot element; one array of size zero, i.e. the (empty) array to the left of the pivot if the pivot is on the left and to the right of the pivot if the pivot is on the right). We also have $Y_3 = \frac{2}{3}(Y_0 + Y_1 + Y_2) = \frac{10}{3}$. We claim that for $N \geq 3$ we have $Y_N = \frac{5(N+1)}{6}$. Clearly this is true for $N = 3$, since $\frac{5(3+1)}{6} = \frac{10}{3}$. Suppose it is true for all $N \leq n$. Then we have

$$Y_{n+1} = \frac{n+2}{n+1}Y_n = \frac{n+2}{n+1} \cdot \frac{5(n+1)}{6} = \frac{5(n+2)}{6},$$

completing our induction. Therefore we have

$$\boxed{Y_0 = 1, Y_1 = 1, Y_2 = 3, Y_N = \frac{5(N+1)}{6} \ \ \forall N \geq 3}.$$