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COS 488
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PS1 - Q1

For this recurrence, we will first multiply both sides by N:

$$NA_N = N + 2\sum_{1\le j\le N} A_{j-1} \qquad \text{for } N > 0 \tag{1}$$

Next subtract the same equation for $N-1$:

$$NA_N - (N-1)A_{N-1} = 1 + 2A_{N-1} \qquad \text{for } N > 1 \tag{2}$$

Rearranging gives

$$NA_N = 1 + (N+1)A_{N-1} \qquad \text{for } N > 1 \tag{3}$$

Now divide the equation by $N(N+1)$ to get:

$$\frac{A_N}{N+1} = \frac{1}{N(N+1)} + \frac{A_{N-1}}{N} \qquad \text{for } N > 1 \tag{4}$$

Iterating gives:

$$\frac{A_N}{N+1} = \sum_{1\le j\le N} \frac{1}{j(j+1)} + \frac{A_1}{2} \qquad \text{for } N > 1 \tag{5}$$

and with $A_1 = 1$ from the original recurrence we have:

$$\frac{A_N}{N+1} = \sum_{1\le j\le N} \frac{1}{j(j+1)} \qquad \text{for } N > 1 \tag{6}$$

Noticing that $\frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1}$, we write this sum as:

$$\sum_{1\le j\le N} \frac{1}{j(j+1)} = \sum_{1\le j\le N} \left(\frac{1}{j} - \frac{1}{j+1}\right) = 1 - \frac{1}{N+1} = \frac{N}{N+1} \tag{7}$$

and so:

$$A_N = N \qquad \text{for } N > 1 \tag{8}$$

Since $A_1 = 1$ from the original recurrence, we have:

$$A_N = N \qquad \text{for } N > 0 \tag{9}$$

The way you've set up the recurrence, adding 2/N is already captured by j = 1 and j = N-1. Without this additive term you should be able to get E_N = (N+1)/3.

Equality holds whenever quicksort is called on an array of size 1. For $N > 1$, one of the two recursive calls will be on an array of size 1 if the pivot element is either the second largest or the second smallest element, which occurs with probability $2/N$. If $N = 1$, then the first and only call to quicksort is one on an array of size 1, so in that case equality holds exactly once. We can see a recursive relationship for the number of expected calls with equality. Let's denote E_N the number of expected equality calls for an array of size N. Then we have that $E_1 = 1, E_0 = 0$ since for an array of size 1 there is precisely 1 call with equality (the initial call), and that:

$$E_N = \frac{2}{N} + \frac{1}{N} \sum_{1 \le j \le N} (E_{j-1} + E_{N-j}) \qquad \text{for } N > 1 \tag{10}$$

We can solve this recurrence in the usual way – first multiply by N and note the symmetry of the sum term:

$$NE_N = 2 + 2 \sum_{1 \le j \le N} E_{j-1} \qquad \text{for } N > 1 \tag{11}$$

Next subtract the same equation for $N - 1$:

$$NE_N - (N-1)E_{N-1} = 2E_{N-1} \qquad \text{for } N > 1 \tag{12}$$

Rearrange terms:

$$NE_N = (N+1)E_{N-1} \qquad \text{for } N > 1 \tag{13}$$

Divide by $N(N+1)$

$$\frac{E_N}{N+1} = \frac{E_{N-1}}{N} \qquad \text{for } N > 1 \tag{14}$$

Iterating gives the base case:

$$\frac{E_N}{N+1} = \frac{E_1}{2} \qquad \text{for } N > 1 \tag{15}$$

and substituting in $E_1 = 1$ gives us:

$$E_N = \frac{N+1}{2} \qquad \text{for } N > 1 \tag{16}$$

Note that for $N = 1$, this formula also gives $E_1 = 1$ so we can write generally:

$$E_N = \frac{N+1}{2} \qquad \text{for } N > 0 \tag{17}$$