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PS1-Q1

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Follow through the steps above to solve the recurrence

$$A_N = 1 + \frac{2}{N} \sum_{1 \leq j \leq N} A_{j-1}$$

for N > 0, with $A_0 = 0$. This is the number of times quicksort is called with hi $\geq$ lo. How often does equality hold, on average?

$$NA_N = N + 2 \sum_{1 \leq j \leq N} A_{j-1}$$

$$NA_N - (N-1)A_{N-1} = N - (N-1) + 2A_{N-1}$$

$$NA_N = (N+1)A_{N-1} + 1$$

$$\frac{A_N}{N+1} = \frac{A_{N-1}}{N} + \frac{1}{N(N+1)}$$

$$\frac{A_N}{N+1} = \sum_{1 \leq k \leq N} \frac{1}{k(k+1)} = \frac{N}{N+1}*$$

$$A_N = N$$

hi=lo when we have a 1-element subarray called. The recurrence representing this situation is

$$A_N = \frac{2}{N} \sum_{1 \leq j \leq N} A_{j-1}$$

since it's the same as the recurrences we've solved but without "adding anything" (as with the "1 +" in the second one) until we hit the base case (when size = 1). Thus we have

$$NA_N = 2 \sum_{1 \leq j \leq N} A_{j-1}$$

$$NA_N - (N-1)A_{N-1} = 2A_{N-1}$$

$$A_N = \frac{N+1}{N} A_{N-1}$$

telescoping, we see that

$$A_N = \frac{N+1}{3} A_2.$$

We note that for A_0 we have 0 calls with hi=lo, for A_1 we have 1, for A_2 we have 1 (one element is chosen as the pivot). (Note that since earlier in the problem we subtract the N - 1 case, we must use the upper of two legitimate base cases. Normally N=1 would work, but we must use 2 for it to work here for the stated reason.) Thus

$$A_N = \frac{N+1}{3}.$$

*We note that the above sum is equal to $\frac{N}{N+1}$.
Proof by induction:
Base case: we note that $1/N(N+1) \rightarrow 1/1(1+1) = 1/(1+1) \rightarrow N/(N+1)$.
Now we assume that the sum to N equals N/(N+1) and show that the sum to $N+1$ equals $\frac{N+1}{(N+1)+1} = \frac{N+1}{N+2}$.

$$\sum_{1 \leq k \leq N+1} \frac{1}{k(k+1)} = \frac{N}{N+1} + \frac{1}{(N+1)(N+2)} = \frac{N(N+2)}{(N+1)(N+2)} + \frac{1}{(N+1)(N+2)}$$

$$= \frac{N^2+2N+1}{(N+1)(N+2)} = \frac{(N+1)^2}{(N+1)(N+2)} = \frac{N+1}{N+2}$$