Homework 1: Exercise 1.14

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5/5

a) The number of times `quicksort` is called with $hi \geqslant lo$ is given by the following closed form, with the base cases $A_0 = 0$ and $A_1 = 1$:

$$\begin{aligned}
A_N &= 1 + \frac{2}{N}\sum_{j=1}^{N} A_{j-1} \\
NA_N &= N + 2\sum_{j=1}^{N} A_{j-1} \\
NA_N - (N-1)A_{N-1} &= 1 + 2A_{N-1} \\
NA_N &= 1 + (N+1)A_{N-1} \\
\frac{A_N}{N+1} &= \frac{1}{N(N+1)} + \frac{A_{N-1}}{N} \\
\frac{A_N}{N+1} &= \frac{1}{N(N+1)} + \frac{1}{(N-1)N} + \cdots + \frac{A_1}{2} \\
\frac{A_N}{N+1} &= \sum_{k=1}^{N} \frac{1}{k(k+1)} = \frac{N}{N+1} \\
A_N &= N
\end{aligned}$$

Note that the last equality holds by induction:

$$\begin{aligned}
\sum_{k=1}^{N} \frac{1}{k(k+1)} &= \frac{1}{N(N+1)} + \sum_{k=1}^{N-1} \frac{1}{k(k+1)} \\
&= \frac{1}{N(N+1)} + \frac{N-1}{N} \\
&= \frac{N}{N+1},
\end{aligned}$$

where the base case for $N = 1$ is trivially true.

b) We observe that $hi = lo$ corresponds to subarrays of size 1. Let B_N denote the expected number of times `quicksort` is called on a subarray of size 1. We have the following recurrence for the number of times equality holds, with a base case of $B_2 = 1$:

$$B_N = \frac{1}{N}\sum_{j=0}^{N-1} B_j B_{N-j-1}.$$

Should be a plus sign!

Calculations similar to the ones above, exploiting symmetry, multiplying by N, and subtracting the equation for $N-1$, give the following closed form:

$$\begin{aligned}
B_N &= \frac{2}{N}\sum_{j=0}^{N-1} B_j \\
NB_N &= 2\sum_{j=0}^{N-1} B_j \\
NB_N - (N-1)B_{N-1} &= 2B_{N-1} \\
NB_N &= (N+1)B_{N-1} \\
\frac{B_N}{N+1} &= \frac{B_{N-1}}{N} = \cdots = \frac{B_2}{3} \\
B_N &= \frac{N+1}{3}.
\end{aligned}$$

Note that the base case $B_2 = 1$ indeed holds. Calling `quicksort` on an array of size 2 always partitions the array into the pivot element, a subarray of size 0, and a subarray of size 1.

Also, note that we have to use B_2 as the base case (as opposed to, say, B_1), since the recurrence stops holding for B_1 in terms of B_0, so our subtraction technique breaks down.

Therefore, of the total of N expected calls to `quicksort` on nonempty subarrays, roughly a third ($\frac{N+1}{3N}$) are on subarrays of size 1.