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COS 488/MAT 474
Problem Set 1, Q1

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AofA Exercise 1.14 Solve the recurrence

$$A_N = 1 + \frac{2}{N}\sum_{j=1}^{N} A_{j-1}$$

with $A_0 = 0$. This is the number of times quicksort is called with hi $\geq$ lo. How often does equality hold, on average?

Solution. For all $N \geq 1$, we have

$$
\begin{aligned}
NA_N &= N + 2\sum_{j=1}^{N} A_{j-1} && \text{(multiply by } N\text{)}\\
NA_N - (N-1)A_{N-1} &= 1 + 2A_{N-1} && \text{(subtract } (N-1)A_{N-1}\text{; holds for all } N \geq 1\text{)}\\
NA_N &= 1 + (N+1)A_{N-1} &&\\
\frac{A_N}{N+1} &= \frac{1}{N(N+1)} + \frac{A_{N-1}}{N} && \text{(divide by } N(N+1)\text{)}\\
&= \sum_{j=1}^{N} \frac{1}{j(j+1)} && \text{(telescope; } A_0 = 0\text{)}\\
&= \sum_{j=1}^{N}\left(1 - \frac{j(j+1)-1}{j(j+1)}\right) &&\\
&= \sum_{j=1}^{N}\left(1 - \frac{(j+1)(j-1)+j}{j(j+1)}\right) &&\\
&= \sum_{j=1}^{N}\left(1 - \frac{j-1}{j} - \frac{1}{j+1}\right) &&\\
&= N - \sum_{j=2}^{N}\frac{j-1}{j} - \sum_{j=2}^{N+1}\frac{1}{j} &&\\
&= N - \sum_{j=1}^{N} 1 - \frac{1}{N+1} &&\\
&= 1 - \frac{1}{N+1} &&\\
&= \frac{N}{N+1}. &&
\end{aligned}
$$

Therefore, we have $A_N = N$ for all $N \geq 0$.

The number of times quicksort is called with hi $=$ lo is equal to the number of subarrays of size 1 encountered. We set up the recurrence as follows. For the initial conditions, we have

$$B_0 = 0, \quad B_1 = 1.$$

For all $N \geq 2$, we can write the recurrence:

$$B_N = \frac{2}{N} \sum_{j=1}^{N} B_{j-1}.$$

From this we calculate

$$B_2 = 1.$$

Now we can solve the recurrence similarly to before, to find B_N when $N \geq 3$.

$$\begin{aligned}
NB_N &= 2 \sum_{j=1}^{N} B_{j-1} && \text{(holds for all } N \geq 2\text{)} \\
NB_N - (N-1)B_{N-1} &= 2B_{N-1} && \text{(holds for all } N \geq 3\text{)} \\
NB_N &= (N+1)B_{N-1} \\
\frac{B_N}{N+1} &= \frac{B_{N-1}}{N} \\
&= \frac{B_2}{3} = \frac{1}{3} \\
\implies B_N &= \frac{N+1}{3} \quad \text{for all } N \geq 3.
\end{aligned}$$

Therefore our final solution is

$$B_N = \begin{cases} 0 & \text{if } N = 0; \\ 1 & \text{if } N = 1; \\ \frac{N+1}{3} & \text{if } N \geq 2. \end{cases}$$