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COS 488/MAT 474
Problem Set 1, Q2

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AofA Exercise 1.16 How many subarrays of size 2 or less are encountered, on the average, when sorting a random file of size N with quicksort?

Solution. For the initial conditions, we have

$$A_0 = 1, \quad A_1 = 1, \quad A_2 = 1 + \sum_{j=1}^{2} A_{j-1} = 3.$$

For all $N \geq 3$, we can write the recurrence:

$$A_N = \frac{2}{N} \sum_{j=1}^{N} B_{j-1}.$$

From this we calculate

$$A_3 = \frac{2}{3}(1 + 1 + 3) = \frac{10}{3}.$$

Now we can solve the recurrence to find A_N when $N \geq 4$:

$$\begin{aligned}
NA_N &= 2\sum_{j=1}^{N} A_{j-1} && \text{(holds for all } N \geq 3\text{)} \\
NA_N - (N-1)A_{N-1} &= 2A_{N-1} && \text{(holds for all } N \geq 4\text{)} \\
NA_N &= (N+1)A_{N-1} \\
\frac{A_N}{N+1} &= \frac{A_{N-1}}{N} \\
&= \frac{A_3}{4} = \frac{10}{12} = \frac{5}{6} \\
\implies A_N &= \frac{5(N+1)}{6} \quad \text{for all } N \geq 4.
\end{aligned}$$

Therefore our final solution is

$$A_N = \begin{cases} 1 & \text{if } N = 0, 1; \\ 3 & \text{if } N = 2; \\ \frac{5(N+1)}{6} & \text{if } N \geq 3. \end{cases}$$

Nice