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COS 488 - Homework 1 - Question 1

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Let $A_0 = 0$, and for all $N > 0$, let

$$A_N = 1 + \frac{2}{N}\sum_{k=1}^{N} A_{k-1}.$$

By multiplying both sides by N, subtracting the same equation for $N-1$, and rearranging, we have the following sequence of equalities for all $N > 0$:

$$\begin{aligned} A_N &= 1 + \frac{2}{N}\sum_{1\le k\le N} A_{k-1} \\ NA_N &= N + 2\sum_{1\le k\le N} A_{k-1} \\ NA_N - (N-1)A_{N-1} &= N - (N-1) + 2A_{N-1} \\ NA_N &= 1 + (N+1)A_{N-1} \end{aligned}$$

(We can get away with the third inequality when $N = 1$ because the second equality happens to be true when $N = 0$.) By dividing both sides by $N(N+1)$ and telescoping, we have the following sequence of equalities for all $N > 0$:

$$\begin{aligned} \frac{A_N}{N+1} &= \frac{1}{N(N+1)} + \frac{A_{N-1}}{N} \\ &= \frac{1}{N(N+1)} + \frac{1}{(N-1)N} + \frac{A_{N-2}}{N-1} \\ &= \sum_{k=1}^{N} \frac{1}{k(k+1)} + \frac{A_0}{1} \\ &= \sum_{k=1}^{N} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= 1 - \frac{1}{N+1} \end{aligned}$$

Therefore, $A_N = N$ for all $N > 0$ (and indeed for all $N \ge 0$).
Now, for all $N \ge 0$, let B_N be the expected value of the number of times that quicksort is called with $hi = lo$ when sorting N items, so that $B_0 = 0$, $B_1 = 1$, $B_2 = 1$, and for all $N > 2$,

$$B_N = \frac{1}{N}\sum_{k=0}^{N-1}(B_k + B_{N-1-k})$$

since $B_k + B_{N-1-k}$ is the expected value of the number of times that quicksort is called with $hi = lo$ when sorting N items assuming the partitioning element is the k^{th} smallest, which has a probability of $\frac{1}{N}$ for each $0 \le k \le N-1$. Then, by applying the symmetry of the sum and repeating the same steps as in the above solution, we have the following sequence of equalities for all $N > 2$:

$$\begin{aligned}
B_N &= \frac{1}{N}\sum_{k=0}^{N-1}(B_k + B_{N-1-1}) \\
B_N &= \frac{2}{N}\sum_{k=0}^{N-1} B_k \\
NB_N &= 2\sum_{k=0}^{N-1} B_k \\
NB_N - (N-1)B_{N-1} &= 2B_{N-1} \\
NB_N &= (N+1)B_{N-1} \\
\frac{B_N}{N+1} &= \frac{B_{N-1}}{N} = \cdots = \frac{B_2}{3} = \frac{1}{3}
\end{aligned}$$

Therefore, $B_0 = 0$, $B_1 = 1$, and $B_N = \frac{N+1}{3}$ for all $N > 1$.