COS 488 - Homework 1 - Question 2

Matt Tyler

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For all $N \geq 0$, let T_N be the expected value of the number of times that quicksort is called with $hi - lo \in \{-1, 0, 1\}$ when sorting N items, so that $T_0 = 1$, $T_1 = 1$, $T_2 = 3$, and for all $N > 2$,

$$T_N = \frac{1}{N} \sum_{k=0}^{N-1} (T_k + T_{N-1-k})$$

since $T_k + T_{N-1-k}$ is the expected value of the number of times that quicksort is called with $hi - lo \in \{-1, 0, 1\}$ when sorting N items assuming the partitioning element is the k^{th} smallest, which has a probability of $\frac{1}{N}$ for each $0 \leq k \leq N - 1$.

Therefore, $T_3 = \frac{10}{3}$, and by applying the symmetry of the sum and repeating the same steps as in the solution to Question 1, we have the following sequence of equalities for all $N > 3$:

$$\begin{aligned}
T_N &= \frac{1}{N} \sum_{k=0}^{N-1} (T_k + T_{N-1-1}) \\
T_N &= \frac{2}{N} \sum_{k=0}^{N-1} T_k \\
NT_N &= 2 \sum_{k=0}^{N-1} T_k \\
NT_N - (N-1)T_{N-1} &= 2T_{N-1} \\
NT_N &= (N+1)T_{N-1} \\
\frac{T_N}{N+1} &= \frac{T_{N-1}}{N} = \cdots = \frac{T_3}{4} = \frac{5}{6}
\end{aligned}$$

Therefore, $T_0 = 1$, $T_1 = 1$, $T_2 = 3$, and $T_N = \frac{5(N+1)}{6}$ for all $N > 2$.