COS 488 Problem Set #1 Problem #1

Tim Ratigan

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Since the program treats empty subarrays at the start of a loop, we will count them once. Since it is told to stop at subarrays of size 1, these will also be counted once. Subarrays of size 2 will recur into one subarray of size 1 and one empty subarray, so including the subarray of size 2 these will involve encountering 3 small subarrays. If S_N denotes the number of small subarrays encountered on average when given a random list of length N, then for $N \geq 2$ we have the following recursion:

$$S_{N+1} = \frac{1}{N+1}\sum_{j=0}^{N}(S_j + S_{N-j}) = \frac{2}{N+1}\sum_{j=0}^{N} S_j$$

This is because the pointer lands at each index with probability $1/(N+1)$ and then proceeds to sort the lists below it and above it, which have lengths j and $N-j$. Since the initial list was random, these lists will also be random so the average number of small subarrays encountered will be $S_j + S_{N-j}$.

$$\begin{aligned} S_{N+1} &= \frac{2}{N+1}\sum_{j=0}^{N} S_j \\ &= \frac{2}{N+1}S_N + \frac{N}{N+1}S_N \\ &= \frac{N+2}{N+1}S_N \end{aligned}$$

Telescoping, this gives that $S_{N+1} = \frac{N+2}{4}S_3$. We have from our recurrence that $S_3 = \frac{2}{3}(1+1+3) = \frac{10}{3}$, so we obtain that, for $N > 2$,

$$S_N = \frac{N+1}{4}\cdot\frac{10}{3} = \frac{5(N+1)}{6}$$