5.5/6

David Luo
COS 488

V.2 Give asymptotic expressions for the number of objects of size N and the number of parts in a random object of size N for the following classes: compositions of 1s, 2s, and 3s, triple surjections, and alignments with no singleton cycles.

The OGF of compositions of 1s, 2s, and 3s is that of a sequence of 1s, 2s, and 3s:

$$C(z) = \frac{1}{1-z-z^2-z^3}$$

We can use the supercritical schema here because $G(z) = z + z^2 + z^3$ has a radius of convergence $\rho = \infty$ and $G(\rho) = \infty$ which is greater than 1:

$$c_N \sim \frac{1}{G'(\lambda)\lambda^{N+1}}$$

We can solve for $G(\lambda) = 1$ to get λ, which is 0.54369:

$$c_N \sim \frac{1}{(1+2\lambda+3\lambda^2)\lambda^{N+1}}$$

$$= 1.61703 \times 0.54369^N$$

-0.5
you forgot to take the reciprocal
should be 0.618418*1.839283^N

Finally, the expected number of G-components in a random C-component of size N is

$$\mu_N \sim \frac{N+1}{\lambda G'(\lambda)} + \frac{G''(\lambda)}{G'(\lambda)^2} - 1$$

$$= \frac{N+1}{\lambda(1+2\lambda+3\lambda^2)} + \frac{2+6\lambda}{(1+2\lambda+3\lambda^2)^2} - 1$$

$$= 0.618418N + 0.213297$$

The class of triple surjections is constructed as

$$S = SEQ(SET_{>2}(z))$$

$$S(z) = \frac{1}{1-\sum_{k=3}^{N}\frac{z^k}{k!}} = \frac{1}{1-e^z+1+z+\frac{z^2}{2}} = \frac{1}{2-e^z+z+\frac{z^2}{2}}$$

We can use the supercritical schema here again, since $G(z)$ converges with radius $\rho = \infty$. We also have the real root of $G(\lambda) = 1$ through solving, $\lambda = 1.56812$ which gives us

$$s_N \sim \frac{1}{G'(\lambda)\lambda^{N+1}} = \frac{1}{(e^\lambda-1-\lambda)\lambda^{N+1}}$$

$$\sim 0.448531 \times 1.56812^{-N-1}$$

And the expected number of G-components in a random S-component of size N is

$$\mu_N \sim \frac{N+1}{\lambda G'(\lambda)} + \frac{G''(\lambda)}{G'(\lambda)^2} - 1 = \frac{N+1}{\lambda(e^\lambda-1-\lambda)} + \frac{e^\lambda-1}{(e^\lambda-1-\lambda)^2} - 1$$

$$\sim .286031N - 0.226758$$

The class of alignments without singleton cycles can be described as

$$A = SEQ(CYC_{>1}(z))$$

$$A(z) = \frac{1}{1 - \log\frac{1}{1-z} + z}$$

Here, $G(z) = \log\frac{1}{1-z} - z$, which converges with a radius of convergence $\rho = 1$. Solving for $G(\lambda) = 1$ yields $\lambda = 0.841406$. Our conditions for supercriticality also all hold. So,

$$a_N \sim \frac{1}{G'(\lambda)\lambda^{N+1}} = \frac{1}{(\frac{1}{1-\lambda} - 1)\lambda^{N+1}}$$

$$\sim 0.224014\, e^{0.172681N}$$

And the expected number of G-components in a random A-component of size N is

$$\mu_N \sim \frac{N+1}{\lambda G'(\lambda)} + \frac{G''(\lambda)}{G'(\lambda)^2} - 1 = \frac{N+1}{\lambda(\frac{1}{1-\lambda} - 1)} + \frac{\frac{1}{(1-\lambda)^2}}{(\frac{1}{1-\lambda} - 1)^2} - 1$$

$$\sim 0.224014N + 0.636514$$