Analytic Combinatorics Web Exercise V.2

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6/6

We start with compositions C of 1s, 2s, and 3s. These are given by the construction $SEQ(Z + Z^2 + Z^3)$, so we have the OGF $C(z) = \frac{1}{1-(z+z^2+z^3)}$. The denominator has root $\lambda \approx 0.544$. Using the formula for asymptotics, we have

$$[z^n]C(z) \sim \frac{-1}{-(3\lambda^2 + 2\lambda + 1)}\left(\frac{1}{\lambda}\right)^n \approx \boxed{0.618 \cdot 1.839^n}.$$

By the formula on slide 57, the expected number of parts in a random element of C of size n is approximately

$$\frac{n+1}{\lambda(3\lambda^2 + 2\lambda + 1)} + \frac{6\lambda + 2}{(3\lambda^2 + 2\lambda + 1)^2} - 1 \approx \boxed{0.618n}.$$

Now we consider triple surjections, which are given by the construction $S = SEQ(SET_{>2}(Z))$, which translates to the EGF $S(z) = \frac{1}{1-\left(e^z - 1 - z - \frac{1}{2}z^2\right)}$. The denominator has root $\lambda \approx 1.568$. Using the formula for asymptotics, we have

$$[z^n]S(z) \sim \frac{-1}{-\lambda\left(e^\lambda - 1 - \lambda\right)}\left(\frac{1}{\lambda}\right)^n \approx 0.286 \cdot 0.638^n.$$

Multiplying by $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ (since we are working with an EGF) gives that the number of triple surjections of size n is approximately

$$0.286\sqrt{2\pi n}\left(\frac{0.638n}{e}\right)^n \approx \boxed{0.717\sqrt{n}(0.235n)^n}.$$

Applying the formula on slide 57, we have that the expected number of parts in a random element of S of size n is approximately

$$\frac{n+1}{\lambda(e^\lambda - 1 - \lambda)} + \frac{e^\lambda - 1}{(e^\lambda - 1 - \lambda)^2} - 1 \approx \boxed{0.286n}.$$

Finally we consider alignments with no singleton cycles, which are given by the construction $A = SEQ(CYC_{>1}(Z))$, which translates to the EGF $A(z) = \frac{1}{1-\left(\ln\frac{1}{1-z} - z\right)} = \frac{1}{1-(-z-\ln(1-z))}$. The denominator has root $\lambda \approx 0.841$. Using the formula for asymptotics, we have

$$[z^n]A(z) \sim \frac{-1}{-\lambda\left(\frac{1}{1-\lambda} - 1\right)}\left(\frac{1}{\lambda}\right)^n \approx 0.224 \cdot 1.188^n.$$

Multiplying by $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ (since we are working with an EGF) gives that the number of alignments with no singleton cycles is approximately

$$0.224\sqrt{2\pi n}\left(\frac{1.188n}{e}\right)^n \approx \boxed{0.562\sqrt{n}(0.437n)^n}.$$

Note that we are dealing with a supercritical sequence class, since as we approach the radius of convergence of $G(z) = -z - \ln(1-z)$ along the positive real axis, we eventually reach a number larger than 1, since we are heading toward a pole. We may therefore use the formula on slide 57, which tells us that the expected number of parts in a random element of A of size n is approximately

$$\frac{n+1}{\lambda\left(\frac{1}{1-\lambda}-1\right)} + \frac{\frac{1}{(1-\lambda)^2}}{\left(\frac{1}{1-\lambda}-1\right)^2} - 1 \approx \boxed{0.224n.}$$