Analytic Combinatorics Homework 10 Question and Answer

Eric Neyman

4/25/2017

Let $f(n)$ be the number of ways to write n as a composition (ordered sum) of odd positive integers.

(a) Compute $f(n)$ for $n = 1, 2, 3, 4, 5$.

(b) Write down a combinatorial construction for the class of compositions composed of odd positive integers.

(c) Translate the combinatorial construction to a generating function. Write the generating function in a closed form (without any infinite sums).

(d) Compute the asymptotics of $f(n)$.

(a) 1 can be written only as 1; 2 as 1+1; 3 as 3 or 1+1+1; 4 as 3+1, 1+3, or 1+1+1+1; and 5 as 5, 3+1+1, 1+3+1, 1+1+3, or 1+1+1+1+1. Thus, we have $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, $f(4) = 3$, and $f(5) = 5$.

(b) Let C be the class of compositions composed of odd positive integers. We have $C = SEQ(Z + Z^3 + Z^5 + \dots)$.

(c) This translates immediately to

$$C(z) = \frac{1}{1 - z - z^3 - z^5 - \dots} = \frac{1}{1 - z\left(1 + z^2 + z^4 + \dots\right)} = \frac{1}{1 - \frac{z}{1-z^2}} = \frac{1 - z^2}{1 - z - z^2}.$$

(d) Solving $1 - z - z^2 = 0$ we find that $z = \frac{-1\pm\sqrt{5}}{2}$, so the pole of $C(z)$ nearest the origin is at $\hat{\phi} = \frac{\sqrt{5}-1}{2}$. Letting $f(z) = 1 - z^2$ and $g(z) = 1 - z - z^2$, we find that

$$\begin{aligned} f(n) = [z^n]C(z) &\sim \frac{-h_1}{\hat{\phi}}\left(\frac{1}{\hat{\phi}}\right)^n = \frac{-f(\hat{\phi})}{\hat{\phi}g'(\hat{\phi})}\left(\frac{1}{\hat{\phi}}\right)^n = \frac{\hat{\phi}^2 - 1}{\hat{\phi}(-2\hat{\phi} - 1)}\left(\frac{1}{\hat{\phi}}\right)^n = \frac{1 - \hat{\phi}^2}{\hat{\phi}(2\hat{\phi} + 1)}\phi^n \\ &= \frac{1 - (1 - \hat{\phi})}{2(1 - \hat{\phi}) + \hat{\phi}}\phi^n = \frac{\hat{\phi}}{2 - \hat{\phi}}\phi^n = \frac{\sqrt{5} - 1}{5 - \sqrt{5}}\phi^n = \frac{(-1 + \sqrt{5})(5 + \sqrt{5})}{20}\phi^n = \frac{4\sqrt{5}}{20}\phi^n = \boxed{\frac{1}{\sqrt{5}}\phi^n}. \end{aligned}$$