COS 488: AC week 3 Q1

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You are a lab TA for an intro Analytic Combinatorics class! John is a freshman who is currently in the class and doing research project. He has derived the generating function for the number of "black and white reversible strings with N-1 beads" to be:

$$G(z) = \frac{-z(3z^2 - 1)}{(2z - 1)(2z^2 - 1)}$$

but he missed the lectures on meromorphic functions! He needs your help finding an asymptotic estimate by computing the residue.
The Professor has also requested that all students, in their research projects, compare their asymptotic estimates to the number of ordered partitions of the integers by finding the ratio of these values. Please derive this formula then find the ratio as well.
(Bonus: A student in the class has asked you to derive the ratio of permutations that contain no perfect-square-length cycles. If you can remember a little bit of analysis, show her how to do so.)

For the first generating function, we see that the unique closest pole the origin $\alpha = 1/2 < 1/\sqrt{2}$. Expanding the bottom and taking a derivative gives $g'(z) = 4z^3 - 2z^2 - 2z + 1$, and note $f(z) = -z(3z^2 - 1)$. Thus the residue is thus

$$h_{-1} = -f(\alpha)/g'(\alpha) = 1/4$$

and the coefficient is then

$$[z^N]G(z) \sim 2^{N-2}$$

(Note that the OEIS entry gives 8256 as the 15th entry, for instance, while we get $2^{13} = 8192$, thus confirming our estimate.)
The class of compositions is:

$$C = SEQ(I)$$

$$C(z) = \frac{1}{1 - \frac{z}{1-z}} = \frac{1-z}{1-2z}$$

$$[z^N]C(z) \sim 2^{N-1}$$

Thus the ratio of John's asymptotic estimate to the number of compositions is exactly 1/2.
(I found that the other one worked out nicely, so I thought it wouldn't hurt to add this.)
The last result is a "bonus" because it requires knowledge of a nontrivial number theory/analysis result, and aside from that it's still quite an easy question. I thought it was kind of a cool result though when I thought of it the other day, so I thought it wouldn't hurt to put here.
Let P_M be the class of all permutations with no cycles of length equal to that of a perfect square.

$$P_M = SET(CYC_{\neg\square}(Z))$$

$$P_M(Z) = \frac{e^{-z-z^4/z-z^9/9-z^{16}/16-\dots}}{1-z}$$

this has a dominant singularity in the form of a pole at 1. Thus the residue is:

$$h_{-1} = -\frac{f(1)}{g'(1)} = e^{-(1+1/4+1/9+1/16+\dots)} = e^{-\pi^2/6}$$

thus

$$[z^N]P_M(z) \sim 1/e^{\pi^2/6} \sim .19$$

It may be better to reform this question in the form of a matching question, maybe add another result with permutations as well. I was going to do that originally, but then I wanted to keep the use of the $\pi^2/6$-related result, but thought it wouldn't be fair to do so without giving a hint to whoever's solving the question.
I got the idea for the first part while browsing OEIS; it corresponds to sequence A005418, and of course my derivation for compositions is from AC05-PoleApps.pdf.