Homework 10: Exercise V.2

Maryam Bahrani (`mbahrani`)

6/6

We will use the transfer theorem for supercritical sequence schema to solve these problems.

1 Compositions of 1s, 2s, and 3s

1.1 Enumeration

A composition can be thought of as a sequence of non-zero integers, where an integer is a sequence of atoms. The class of combinations of 1s, 2s, and 3s can therefore be specified as

$$\mathcal{C} = \text{SEQ}(\mathcal{G})$$
$$\mathcal{G} = \text{SEQ}_{1,2,3}(\mathcal{Z})$$
$$G(z) = z + z^2 + z^3.$$

The smallest real root of $G(z) = 1$ is $\lambda = 0.543689$. The coefficients can be extracted via the transfer theorem for supercritical sequence classes:

$$[z^n]C(z) \sim \frac{1}{G'(\lambda)}\frac{1}{\lambda^{n+1}} = \frac{1}{1 + 2\lambda + 3\lambda^2}\frac{1}{\lambda^{n+1}} = \boxed{.618 \cdot 1.839^n}.$$

1.2 Number of Parts

Using the Corollary presented in lecture on the number of components in supercritical sequence classes, the average number of parts μ_n is

$$\mu_n \sim \frac{n+1}{\lambda G'(\lambda)} + \frac{G''(\lambda)}{G'(\lambda)^2} - 1 = \boxed{0.618n + 0.213}.$$

2 Triple surjections

2.1 Enumeration

Triple surjections can be specified as a sequence of sets of size at least 3:

$$\mathcal{S} = \text{SEQ}(\mathcal{G})$$
$$\mathcal{G} = \text{SET}_{\geqslant 3}(\mathcal{Z})$$
$$G(z) = e^z - 1 - z - \frac{z^2}{2}.$$

The smallest real root of $G(z) = 1$ is $\lambda = 1.56812$. The coefficients can be extracted via the transfer theorem for supercritical sequence classes:

$$[z^n]S(z) \sim \frac{1}{G'(\lambda)}\frac{1}{\lambda^{n+1}} = \frac{1}{e^\lambda - 1 - \lambda}\frac{1}{\lambda^{n+1}} = 0.286 \cdot 0.638^n.$$

The number of triple surjections can be derived via Stirling's approximation:

$$n![z^n]S(z) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot 0.286 \cdot 0.638^n = \boxed{0.717\sqrt{n}(0.235n)^n}.$$

2.2 Number of Parts

Using the Corollary presented in lecture on the number of components in supercritical sequence classes, the average number of parts μ_n in triple surjections is

$$\mu_n \sim \frac{n+1}{\lambda G'(\lambda)} + \frac{G''(\lambda)}{G'(\lambda)^2} - 1 = \boxed{0.286n + 0.050}.$$

3 Alignments with no singleton cycles

3.1 Enumeration

Alignments are sequences of cycles, so allignments with no singleton cycles can be specified as

$$\begin{aligned} \mathcal{A} &= \textsc{Seq}\,(\mathcal{G}) \\ \mathcal{G} &= \textsc{Cyc}_{\geqslant 1}\,(\mathcal{Z}) \\ G(z) &= \ln\frac{1}{1-z} - z. \end{aligned}$$

The smallest real root of $G(z) = 1$ is $\lambda = 0.841406$. The coefficients can be extracted via the transfer theorem for supercritical sequence classes:

$$[z^n]A(z) \sim \frac{1}{G'(\lambda)}\frac{1}{\lambda^{n+1}} = \frac{1-\lambda}{\lambda}\frac{1}{\lambda^{n+1}} = 0.224 \cdot 1.188^n.$$

The number of alignments with no singelton cycles can be derived via Stirling's approximation:

$$n![z^n]A(z) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot 0.224 \cdot 1.188^n = \boxed{0.561\sqrt{n}(0.437n)^n}.$$

3.2 Number of Parts

Using the Corollary presented in lecture on the number of components in supercritical sequence classes, the average number of parts μ_n in alignments with no singelton cycles is

$$\mu_n \sim \frac{n+1}{\lambda G'(\lambda)} + \frac{G''(\lambda)}{G'(\lambda)^2} - 1 = \boxed{0.224n + 0.637}$$