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COS 488/MAT 474
Problem Set 10, Q1

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AC Web Exercise V.1 Give an asymptotic expression for the number of strings that do not contain the pattern 0000000001. Do the same for 0101010101.

Solution. As we have shown earlier in the course, the OGF for bitstrings not containing the k-bit pattern p is

$$B_p(z) = \frac{c_p(z)}{z^k + (1-2z)c_p(z)},$$

where $c_p(z)$ is the autocorrelation polynomial for the pattern.

Let p_1 be the 10-bit pattern 0000000001. Its autocorrelation polynomial is $c_{p_1}(z) = 1$, so the OGF for bitstrings not containing p_1 is

$$B_{p_1}(z) = \frac{1}{z^{10} - 2z + 1}.$$

This is a meromorphic function. The smallest real root of the denominator is $\alpha_1 \doteq 0.500493$. Its reciprocal is $\beta_1 \doteq 1.998030$. The constant term is

$$c_1 = \frac{-1}{\alpha_1(10\alpha_1^9 - 2)} \doteq 1.00896.$$

Therefore, by the transfer theorem for meromorphic functions, the number of bitstrings of length N that do not contain the pattern 0000000001 is asymptotic to

$$[z^N]B_{p_1}(z) \sim c_1\beta_1^N, \qquad c_1 \doteq 1.00896,\ \beta_1 \doteq 1.998030.$$

Let p_2 be the 10-bit pattern 0101010101. Its autocorrelation polynomial is $c_{p_2}(z) = 1 + z^2 + z^4 + z^6 + z^8$, so the OGF for bitstrings not containing p_2 is

$$B_{p_2}(z) = \frac{1 + z^2 + z^4 + z^6 + z^8}{z^{10} + (1-2z)(1 + z^2 + z^4 + z^6 + z^8)}.$$

This is a meromorphic function. The smallest real root of the denominator is $\alpha_2 \doteq 0.500369$. Its reciprocal is $\beta_2 \doteq 1.998525$. The constant term is

$$c_2 = \frac{-(1 + \alpha_2^2 + \alpha_2^4 + \alpha_2^6 + \alpha_2^8)}{\alpha_2(10\alpha_2^9 - 2(1 + \alpha_2^2 + \alpha_2^4 + \alpha_2^6 + \alpha_2^8) + (1-2\alpha_2)(2\alpha_2 + 4\alpha_2^3 + 6\alpha_2^5 + 8\alpha_2^7))} \doteq 1.0062.$$

Therefore, by the transfer theorem for meromorphic functions, the number of bitstrings of length N that do not contain the pattern 0101010101 is asymptotic to

$$[z^N]B_{p_2}(z) \sim c_2\beta_2^N, \qquad c_2 \doteq 1.0062,\ \beta_2 \doteq 1.998525.$$