Miranda Moore
COS 488/MAT 474
Problem Set 10, Q2

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AC Web Exercise V.2 Give asymptotic expressions for the number of objects of size N and the number of parts in a random object of size N for the following classes:

(a) Compositions of 1s, 2s, and 3s

(b) Triple surjections

(c) Alignments with no singleton cycles

Solution. (a) Compositions (ordered sums) of 1s, 2s, and 3s have the construction

$$\mathcal{C} = SEQ(\mathcal{I}_{1,2,3}).$$

This is a strongly aperiodic supercritical sequence class, with $G(z) = I_{1,2,3}(z) = z + z^2 + z^3$. The smallest real root of $G(z) = 1$ is $\lambda \doteq 0.54369$. Therefore, by the transfer theorem for supercritical sequence classes, the number of compositions of N composed of 1s, 2s, and 3s is asymptotic to

$$[z^N]C(z) \sim c\left(\frac{1}{\lambda^{N+1}}\right),$$

where

$$c = \frac{1}{G'(\lambda)} = \frac{1}{1 + 2\lambda + 3\lambda^2} \doteq 0.33623, \qquad \lambda \doteq 0.54369.$$

By the corollary, the expected number of components is asymptotic to

$$\mu_N \sim \frac{N+1}{\lambda G'(\lambda)} + \frac{G''(\lambda)}{G'(\lambda)^2} - 1 = \frac{N+1}{\lambda(1 + 2\lambda + 3\lambda^2)} + \frac{2 + 6\lambda}{(1 + 2\lambda + 3\lambda^2)^2} - 1 \doteq 0.618N + 0.213.$$

(b) Triple surjections (surjections where every element in the range is hit at least 3 times) have the construction

$$\mathcal{R} = SEQ(SET_{\geq 3}(\mathcal{Z})).$$

Should be e^z - (1+z+z^2/2)?

This is a strongly aperiodic supercritical sequence class, with $G(z) = e^z - z^2 - z - 1$. The smallest real root of $G(z) = 1$ is $\lambda \doteq 2.20412$. Therefore, by the transfer theorem for supercritical sequence classes, the number of triple surjections of length N is asymptotic to

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$$N![z^N]R(z) \sim c\left(\frac{1}{\lambda^{N+1}}\right)N!,$$

where

$$c = \frac{1}{G'(\lambda)} = \frac{1}{e^\lambda - 2\lambda - 1} \doteq 0.27367, \qquad \lambda \doteq 2.20412.$$

By the corollary, the expected number of components is asymptotic to

$$\mu_N \sim \frac{N+1}{\lambda G'(\lambda)} + \frac{G''(\lambda)}{G'(\lambda)^2} - 1 = \frac{N+1}{\lambda(e^\lambda - 2\lambda - 1)} + \frac{e^\lambda - 2}{(e^\lambda - 2\lambda - 1)^2} - 1 \doteq 0.124N - 0.347.$$

(c) Alignments (sequences of cycles) with no singleton cycles have the construction

$$\mathcal{O} = SEQ(CYC_{\geq 2}(\mathcal{Z})).$$

This is a strongly aperiodic supercritical sequence class, with $G(z) = \log \frac{1}{1-z} - z$. The smallest real root of $G(z) = 1$ is $\lambda \doteq 0.8414$. Therefore, by the transfer theorem for supercritical sequence classes, the number of alignments of N atoms with no singleton cycles is asymptotic to

$$N![z^N]O(z) \sim c\left(\frac{1}{\lambda^{N+1}}\right)N!,$$

where

$$c = \frac{1}{G'(\lambda)} = \frac{1}{\frac{1}{1-\lambda} - 1} \doteq 0.1885, \qquad \lambda \doteq 0.8414.$$

By the corollary, the expected number of components is asymptotic to

$$\mu_N \sim \frac{N+1}{\lambda G'(\lambda)} + \frac{G''(\lambda)}{G'(\lambda)^2} - 1 = \frac{N+1}{\lambda(\frac{1}{1-\lambda} - 1)} + \frac{\frac{1}{(1-\lambda)^2}}{(\frac{1}{1-\lambda} - 1)^2} - 1 \doteq 0.224N - 0.637.$$