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COS 488/MAT 474
Problem Set 10, Q&A
Applications of Meromorphic Asymptotics

Question Let k be a positive integer. Find an asymptotic approximation for the number of words of length N that are M-surjections for any M such that $M \equiv 0 \mod k$.

Solution. This class of surjections has the construction

$$\mathcal{R}_k = SEQ_{\equiv 0 \bmod k}(SET_{\geq 0}(\mathcal{Z})),$$

i.e., the sequences can be of length $0, k, 2k, 3k, \ldots$, and so on. The generating function for $SEQ_{\equiv 0 \bmod k}(\mathcal{Z})$ is

$$\frac{1}{1 - z^k} = 1 + z^k + z^{2k} + z^{3k} + \cdots,$$

so the EGF for $\mathcal{R}_k$ is

$$R_k(z) = \frac{1}{1 - (e^z - 1)^k}.$$

The poles of this function are located wherever $e^z - 1$ is a kth root of unity; there are infinitely many poles because the complex logarithm has infinitely many branches! However, there is only one real solution, which occurs where $e^z - 1 = 1 \;\Rightarrow\; z = \ln 2$. This is also the unique pole of smallest modulus. Plugging into the transfer theorem for meromorphic functions, we have

$$\alpha = \ln 2, \qquad \beta = \frac{1}{\ln 2}, \qquad c = \frac{-1}{\alpha \cdot -k(e^\alpha - 1)e^\alpha} = \frac{1}{2k \ln 2}.$$

Therefore,

$$[z^N]R_k(z) \sim \frac{1}{2k(\ln 2)^{N+1}}$$

and the number of words of length N in $\mathcal{R}_k$ is $\sim \frac{N!}{2k(\ln 2)^{N+1}}$.

This makes sense! Recall that $[z^N]R(z) \sim \frac{1}{2(\ln 2)^{N+1}}$, where $\mathcal{R}$ is the class of all surjections. We'd expect that the number of surjections of length N, whose number of unique letters is a multiple of k, should be about $\frac{1}{k}$ of the total number of surjections of length N. This is exactly what we have shown.