COS 488 - Homework 10 - Question & Answer

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1 Question

Inspiration for this question came from the calculation of the number of connected components of mappings in the second Analytic Combinatorics Lecture.

1. Given a positive integer N, let $f_N(z) = \frac{e^{Nz}}{1-z}$. Find an asymptotic expression for the coefficients of $f_N(z)$.
2. Calculate the coefficient of z^{N-1} in $f_N(z)$ explicitly in terms of the Ramanujan Q-function

$$Q(N) = \sum_{k=1}^{N} \frac{N!}{N^k (N-k)!}.$$

3. Consider the fact that $Q(N) \sim \sqrt{\frac{\pi N}{2}}$. How does this compare to the results computed in parts 1 and 2? What went wrong?

2 Answer

1. Since $f_N(z)$ is meromorphic when $|z| \leq 2$ and analytic both when $z = 0$ and when $|z| = 2$, we may use the AC transfer theorem. The function $f_N(z)$ has a unique closest pole to the origin at $z = 1$, and it has order 1, so we have
$$[z^n]f_N(z) \sim -\frac{e^{1\times N}}{-1}1^N = e^N.$$

2. We have
$$\begin{aligned}
[z^{N-1}]f_N(z) &= [z^{N-1}]\left(\left(\sum_{k=0}^{\infty}\frac{N^k z^k}{k!}\right)\left(\sum_{k=0}^{\infty}z^k\right)\right)\\
&= \sum_{k=0}^{N-1}\frac{N^k}{k!}\\
&= \sum_{k=1}^{N}\frac{N^{N-k}}{(N-k)!}\\
&= \frac{N^N}{N!}\sum_{k=1}^{N}\frac{N!}{N^k(N-k)!}\\
&= \frac{N^N}{N!}Q(N).
\end{aligned}$$

3. From part 2, we have by Stirling's approximation that
$$[z^{N-1}]f_N(z) \sim \frac{e^N}{\sqrt{2\pi N}}Q(N) \sim \frac{e^N}{2}.$$
However, by part 1, we have $[z^{N-1}]f_N(z) \sim e^N$. These two answers are not the same. The problem arises from the fact that we are not holding the same things constant in the two approximations. In the approximation from part 1, we are holding the N that appears in $f_N(z)$ as a constant, and allowing the N that appears in $[z^{N-1}]$ to vary. In the approximation from part 2, however, both instances of the variable N vary simultaneously. In other words, the approximation from part 1 computes the asymptotic expression "vertically," while the approximation from part 2 computes the asymptotic expression "diagonally." The purpose of this problem is not only to show an application of the AC transfer theorem, but also to show the importance of being careful about what is fixed and what is not in asymptotic calculations.