COS 488 Problem Set #10 Question #1

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For the pattern 0000000001, the autocorrelation polynomial is simply $c(z) = 1$ since any nontrivial displacement will compare one of the zeros to the last 1. Hence, by Theorem 8.3, we have that the EGF for bitstrings not containing this pattern is $f(z) = \frac{1}{z^{10}-2z+1}$. Note that as $P \to \infty$, any roots for $|z| < 1$ tend to $1 - 2z = 0 \implies z = 1/2$, so if we let $z = 1/2 + \epsilon$ then

$$\begin{aligned} z^{10} - 2z + 1 &= 0 \\ (1/2+\epsilon)^{10} - 2(1/2+\epsilon) + 1 &= 0 \\ 2^{-10} + 10\epsilon \cdot 2^{-9} - 2\epsilon &\approx 0 \\ 2^{-10} &\approx \left(2 + \frac{5}{2^8}\right)\epsilon \\ \frac{1}{2^{11} + 5 \cdot 2^2} &\approx \epsilon \end{aligned}$$

Therefore we have $z \approx .50048$ (in reality by using computer algebra packages we can confirm it to be about .500493). Differentiating the polynomial, we have $10z^9 - 2 = 0$, so we can see that this is not a multiple root and therefore a simple pole of the generating function. By applying the transfer theorem, we obtain $[z^n]f(z) \sim C \cdot 1.998^n$ where $C = \frac{-1.998}{10(.500493)^9 - 2} \approx 1.00896$.

For the pattern 0101010101, the autocorrelation polynomial is $c(z) = 1 + z^2 + z^4 + z^6 + z^8$ upon inspection. By Theorem 8.3, the EGF for the bitstrings is $\frac{1+z^2+z^4+z^6+z^8}{z^{10}+(1-2z)(1+z^2+z^4+z^6+z^8)} = \frac{1+z^2+z^4+z^6+z^8}{z^{10}-2z^9+z^8-2z^7+z^6-2z^5+z^4-2z^3+z^2-2z+1}$. Once again, as $P \to \infty$ polynomials with the same autocorrelation coefficient will have a real root about 1/2, so we can let $z = 1/2 + \epsilon$ and then we have

$$\begin{aligned} 2^{-10} + 10 \cdot 2^{-9}\epsilon - 2\epsilon(1 + 2^{-2} + 2^{-4} + 2^{-6} + 2^{-8}) &\approx 0 \\ 2^{-10} &\approx \epsilon\left(5 \cdot 2^{-8} + 2\frac{1-2^{-10}}{1-2^{-2}}\right) \\ .0003639 &\approx \epsilon \end{aligned}$$

From computer algebra, we know that the root is actually about .500369. The constant in the transfer theorem evaluates to $\frac{-1.99852 \cdot 1.33268}{-2.64699} \approx 1.0062$, since by the same logic as above the pole is simple, so $[z^n]f(z) \sim 1.0062 \cdot .500369^n$.

You want the reciprocal of the root of smallest modulus which is 1/0.500369 which is about 1.998525, as a sanity check your current coefficients tend to 0

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