COS 488 Problem Set #10 Question #2

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Compositions of 1s, 2s, and 3s are given by $SEQ(\mathcal{Z}+\mathcal{Z}^2+\mathcal{Z}^3)$, which gives the generating function $\frac{1}{1-z-z^2-z^3}$. We then obtain that there is a simple pole at $z \approx 0.543689$, and so by the transfer theorem the coefficients are asymptotic to $.61842 \cdot 1.83929^n$.'As noted in the lecture slides, we can compute the expected number of components in a composition of size N by letting $G(z) = z + z^2 + z^3$ since this is entire and therefore trivially supercritical. Then $\mu_N \sim \frac{N+1}{z'(1+2z'+z'^2)} + \frac{2+6z'}{(1+2z'+3z'^2)^2} - 1 \approx .61842(N+1) - .504119$ where z' is the pole found earlier.

The combinatorial class of triple surjections is enumerated by $SEQ(SET_{>2}(Z))$ since each letter must appear at least 3 times. $SET_{>2}(Z)$ corresponds to $e^z - z^2/2 - z - 1$ and so the EGF is $\frac{1}{2+z+z^2/2-e^z}$. This has a singularity at $z' \approx 1.56812$. This is clearly simple, so the constant term in the transfer theorem is $\frac{-1/z'}{1+z'-e^{z'}} \approx .286031$, and hence we have that the number of triple surjections is asymptotic to $.286031 \cdot .637706^n n!$. Moreover, $e^z - z^2/2 - z - 1$ is entire so we can apply the theorem for obtaining the number of components. This gives us

$$\mu_N \sim \frac{N+1}{z'(e^{z'} - z' - 1)} + \frac{e^{z'} - 1}{(e^{z'} - z' - 1)^2} - 1 \approx .286031(N+1) - .235994$$

The number of alignments with no singleton cycles is given by $SEQ(CYC_{>1}(Z))$ which corresponds to $\frac{1}{1-\log\frac{1}{1-z}+z}$. Solving for $1 + z + \log(1-z) = 0$ gives $z \approx .841406$. The derivative of this function is $1 - \frac{1}{1-z}$, so we obtain from the transfer theorem (since this is clearly a simple pole) that the constant is $\frac{-1/z'}{1-\frac{1}{1-z'}} \approx .224015$, so the number of such alignments is asymptotic to $.224015 \cdot .841406^n n!$. $\log\frac{1}{1-z} - z$ clearly has a radius of convergence of 1, and when $z = 1-\epsilon$ this evaluates to $\epsilon - 1 - \log\epsilon$, so if $\epsilon < e^{-2}$ this is certainly greater than 1 and therefore this is supercritical. Hence, the expected number of components is

$$\mu_N \sim \frac{N+1}{z'((1-z')^{-1} - 1)} + \frac{(1-z')^{-2}}{((1-z')^{-1} - 1)^2} - 1 \approx .224015(N+1) + .412502$$