COS 488 Problem Set #10 Test Example Question

Tim Ratigan

May 4, 2017

Question

If $F = CYC(G)$, find an asymptotic expression for the number of G-components in a random F-component of size n.

Answer

We follow the method suggested in the lectures. If we define $B(z, u) = \log \frac{1}{1-uG(z)}$, then $\mu_n = \frac{1}{f_n}[z^n]\frac{\partial}{\partial u}B(z, u)|_{u=1}$. To calculate f_n, we note $F(z) = \log \frac{1}{1-G(z)}$, so $F'(z) = \frac{G'(z)}{1-G(z)}$. By the transfer theorem for meromorphic functions, if λ is the real root of $G - 1$ in its radius of convergence, then $[z^n]F'(z) \sim \lambda^{-n}\frac{G'(\lambda)}{\lambda G'(\lambda)} = \lambda^{-n-1}$. Integrating, this implies $f_n = [z^n]F(z) \sim \frac{1}{n}\lambda^{-n}$. Meanwhile, $[z^n]\frac{\partial}{\partial u}B(z, u)|_{u=1} = [z^n]\frac{G(z)}{1-G(z)} \sim \lambda^{-n}\frac{G(\lambda)}{\lambda G'(\lambda)}$. As a result, $\mu_n \sim \frac{nG(\lambda)}{\lambda G'(\lambda)}$.