Analytic Combinatorics Web Exercise VI.1

5/5

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We have $D(z) = z + zD(z)^2$, so via the quadratic formula we have $D(z) = \frac{1\pm\sqrt{1-4z^2}}{2z}$. This must converge as z approaches 0, so we want the numerator evaluated at $z = 0$ to be zero; thus, the minus sign is the appropriate one. Similarly, $E(z) = \frac{1-\sqrt{1-4z^2}}{2z}$. Thus, we have

$$S(z) = 1 + z(D(z) + E(z))S(z) = 1 + (1 - \sqrt{1-4z^2})S(z)$$
$$S(z) = \frac{1}{\sqrt{1-4z^2}} = (1-4z^2)^{\frac{-1}{2}}.$$

Applying the standard function scale to $(1-z)^{\frac{-1}{2}}$, we find that

$$[z^N](1-z)^{\frac{-1}{2}} \sim \frac{N^{\frac{-1}{2}}}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi N}}.$$

Now, if we substitute $4z^2$ for z, we find

$$[z^{2N}]S(z) \sim \boxed{\frac{4^N}{\sqrt{\pi N}}}$$

(and the odd terms are all zero).