Analytic Combinatorics Web Exercise VI.2

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Let T be the class of trees where every node has 0, 2, or 3 children. A tree in T is a root together with a sequence of 0, 2, or 3 trees in T. (We define T to not include the empty tree to make this construction work; it does not affect the asymptotics.) Thus, we have $T = \cdot \times SEQ_{0,2,3}(T)$. This translates to the OGF equation $T(z) = z(1 + T(z)^2 + T(z)^3)$. Then $\phi(u)$ as defined on Slide 40 is $1 + u^2 + u^3$, which has nonnegative coefficients and does not have the form $\phi_0 + \phi_1 u$ for constants ϕ_0, ϕ_1; is analytic at 0 with $\phi(0) \neq 0$; and the equation $\phi(\lambda) = \lambda\phi'(\lambda)$ has a unique positive real root $\lambda \approx 0.657$. (The equation for λ is $1 + \lambda^2 + \lambda^3 = \lambda(2\lambda + 3\lambda^2)$, which has the stated unique positive real root.) Therefore, T is an invertible tree class. Applying the formula on Slide 41, we have

$$[z^N]T(z) \sim \frac{1}{\sqrt{2\pi\frac{2+6\lambda}{1+\lambda^2+\lambda^3}}}(2\lambda + 3\lambda^2)^N N^{\frac{-3}{2}} \approx \boxed{0.214 \cdot 2.611^N \cdot N^{\frac{-3}{2}}}.$$

Under an efficient way of mapping these trees to bit strings, one can represent a tree of size N in approximately $\lg([z^N]T(z)) \approx N\lg(2.611) - \frac{3}{2}\lg(N) \approx \boxed{1.384N - 1.5\lg(N)}$ bits.