Analytic Combinatorics Web Exercise VII.1

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Let S be the class of bracketings. We have the generating function $S(z) = z + \frac{S(z)^2}{1-S(z)}$ (Flajolet 69), so we can write $S(z) = \Phi(z, S(z))$, where $\Phi(z,w) = z + \frac{w^2}{1-w}$. We verify that S is a smooth implicit-function tree-like class by using the conditions in Definition VII.4 (Flajolet 467).

We can write $\Phi(z,w) = z + w^2 + w^3 + \dots$. Clearly the first condition is satisfied: take R to be anything (say 10) and $S = 1$; then $\Phi(z,w)$ converges when $|z| < R$ and $|w| < S$. The second condition is also satisfied: all coefficients are nonnegative; the $(0,0)$-coefficient is 0; the $(0,1)$-coefficient is $0 \neq 1$; and the $(0,2)$-coefficient is $1 > 0$. Finally, the third condition is satisfied: take $r = 3 - 2\sqrt{2}$ and $s = 1 - \frac{\sqrt{2}}{2} < 1$. Then

$$\Phi(r,s) = r + \frac{s^2}{1-s} = 3 - 2\sqrt{2} + \frac{\frac{3-2\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 3 - 2\sqrt{2} + \frac{3\sqrt{2}-4}{2} = 1 - \frac{\sqrt{2}}{2}.$$

Also, we have

$$\Phi_w(z,w) = \frac{2w(1-w) + w^2}{(1-w)^2} = \frac{2w - w^2}{(1-w)^2},$$

so

$$\Phi_w(r,s) = \frac{2s - s^2}{(1-s)^2} = \frac{2 - \sqrt{2} - \frac{3-2\sqrt{2}}{2}}{\frac{1}{2}} = 4 - 2\sqrt{2} - (3 - 2\sqrt{2}) = 1.$$

Therefore, S is indeed a smooth implicit-function tree-like class. We may therefore use the theorem on Slide 41. We have

$$\Phi_{ww}(r,s) = \frac{(2-2w)(1-w)^2 + (2w - w^2) \cdot 2(1-w)}{(1-w)^4} = \frac{2}{(1-w)^3},$$

so

$$\alpha = \sqrt{\frac{2(3-2\sqrt{2})}{\frac{2}{\left(\frac{\sqrt{2}}{2}\right)^3}}} = \sqrt{\frac{2(3-2\sqrt{2})}{4\sqrt{2}}} = \frac{\sqrt{2}-1}{2^{\frac{3}{4}}}.$$

Noting that $\frac{1}{r} = \frac{1}{3-2\sqrt{2}} = 3 + 2\sqrt{2}$, we have

$$[z^N]S(z) \sim \frac{\sqrt{2}-1}{2^{\frac{7}{4}}\sqrt{\pi}}(3+2\sqrt{2})^N N^{\frac{-3}{2}} \approx \boxed{0.069 \cdot 5.828^N N^{\frac{-3}{2}}}.$$