Analytic Combinatorics Homework 11 Question and Answer

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A *k-ary tree* is defined the same way as a binary tree, except that each node has k subtrees, rather than 2 subtrees.

(a) Write down a construction and generating function for k-ary sub-trees (where the size of a tree is the number of internal nodes).

(b) Find an asymptotic (i.e. as $N \to \infty$) approximation for the number of k-ary trees with N internal nodes, as a function of N and k.

(c) Assume that k is large. Simplify your asymptotic approximation.

(d) Find a constant c such that if $k \approx cN$, then as N approaches infinity, the number of k-ary trees of size N is approximately the number of Cayley trees of size N. By this we mean that the ratio of the number of Cayley trees of size N to the number of k-ary trees of size N is polynomial, rather than negligible or exponential.

(a) A k-ary tree is a node together with k things, each of which is either empty or a k-ary tree. Letting T be the class of k-ary trees, we have

$$T = \cdot \times (E + B)^k,$$

where the exponent ^{k} represents the Cartesian product $(E + B)$ taken k times. The corresponding generating function is therefore

$$T(z) = z(1 + T(z))^k.$$

(b) We have $T(z) = z\phi(T(Z))$, where $\phi(u) = (1 + u)^k$. We have $\phi'(u) = k(1 + u)^{k-1}$ and $\phi''(u) = k(k-1)u^{k-2}$. We see that the equation $\phi(u) = u\phi'(u)$ has a unique positive real root: we have

$$\begin{aligned}(1 + \lambda)^k &= k\lambda(1 + \lambda)^{k-1} \\ 1 + \lambda &= k\lambda \\ \lambda &= \frac{1}{k-1},\end{aligned}$$

where our first step, dividing by $(1 + \lambda)^{k-1}$, is justified, since the zero there is just $\lambda = -1$ and we are looking for positive roots. In particular, we have $\phi'(\lambda) = k\left(1 + \frac{1}{k-1}\right)^{k-1}$ and $\frac{\phi''(\lambda)}{\phi(\lambda)} = \frac{k(k-1)}{(1+\lambda)^2} = \frac{(k-1)^3}{k}$. Using the theorem of Lecture 6 Slide 41, we have

$$[z^N]T(z) \sim \frac{1}{\sqrt{2\pi\frac{(k-1)^3}{k}}}\left(k\left(1 + \frac{1}{k-1}\right)^{k-1}\right)^N N^{\frac{-3}{2}} = \sqrt{\frac{k}{2\pi(k-1)^3}}\left(\frac{k^k}{(k-1)^{k-1}}\right)^N N^{\frac{-3}{2}}.$$

(c) If k is large, we have $\frac{k}{(k-1)^3} \approx \frac{1}{k}$ and $\left(1 + \frac{1}{k-1}\right)^{k-1} \approx e$. Thus, our formula above simplifies to

$$\boxed{\frac{1}{\sqrt{2\pi}}k^{N-1}e^N N^{\frac{-3}{2}}}.$$

(d) The number of Cayley trees of size N is N^{N-1}. Plugging in cN to the formula above, we notice that the constant c we need to cancel out the e^N is $\boxed{c = \frac{1}{e}}$.