COS 488: AC week 5 QA1

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In lecture AC07, we found an asymptotic estimate for the class of all bracketings. These are also known as the "super-Catalan numbers" and have OGF

$$G(z) = \frac{2}{1 + z + \sqrt{1 - 6z + z^2}}$$

This exact GF is derived in the main textbook, with the result on page 475. It has an error and should be multiplied by $2^{\cdot 25}$ as noted in OEIS A001003.
We will derive F(z), the generating function for F, the class representing "the number of lattice paths from (0,0) to (x,y) that use the step set (0,1),(1,0),(2,0),(3,0),... and never passes below y=x." Note that this has a clear relation to the Catalan numbers, since they normally count the number of paths from (0,0) to (x,x) with exactly n steps up and n right, without passing below the line y=x.
Note that a recursive definition for the terms of the super-catalan numbers is:

$$b_{n+1} = -b_n + 2\sum_{k=0}^{n} b_k b_{n-k}$$

We supply that a recursive formula for the terms of F is

$$a_n = \frac{b_n + b_{n+1}}{2}$$

First: use the recurrence relations given to derive the necessary operation(s) to obtain F(z).
Second: obtain F(z), and simplify if possible.
Third: obtain an asymptotic approximation for the terms of F(z). (Hint: use Asymptotics of implicit tree-like classes with $\alpha \approx 4.901$.)

Solution:
Note that I got the recursive formulas from OEIS entries A001003 and A01068 I used a shifted version of one of them (the b_n one, but the algebra is confirmed by the entry in A010683). First: We plug in the recursive definition b_n into the available terms of a_n:

$$a_n = \frac{b_n + b_{n+1}}{2} = \frac{-b_{n-1} + 2\sum_{k=0}^{n-1} b_k b_{n-k} - b_n + 2\sum_{k=0}^{n} b_k b_{n-k}}{2} = \sum_{k=0}^{n} b_k b_{n-k} + \sum_{k=0}^{n-1} b_k b_{n-k} - \frac{b_{n-1} + b_n}{2}$$

We note that iterating the above substitution will cause the expression to telescope (the terms may alternate positive and negative, but they cancel out all the same) leaving just:

$$a_n = \sum_{k=0}^{n} b_k b_{n-k}$$

This may actually be a bit annoying to show that the terms work out exactly to give this in the edge cases, and besides, I found a much simpler way to do it after writing the above. We can simply rearrange the original expression I gave for b_{n+1} to immediately get the desired result:

$$\frac{b_{n+1} + b_n}{2} = \sum_{k=1}^{n} b_k b_{n-k} = a_n$$

Second: after deriving this, we notice that this is just a convolution of our generating function G(z) with itself, as this is the exact definition of such a convolution! Thus we have:

$$F(z) = G(z)^2 = (\frac{2}{1 + z + \sqrt{1 - 6z + z^2}})^2$$

which after some simplification, gives

$$F(z) = \frac{(1-z)^2 - (z+1)\sqrt{z^2 - 6z + 1}}{8z^2}$$

(confirmed with the OEIS entries above).
Third: we look for the singularities in our above expression. The roots of $x^2 - 6x + 1$ are $x = 3 \pm 2\sqrt{2}$. Perhaps it would have been nice to add this to the original question as a hint to the students that they are on the right track and to avoid using the quadratic formula, even though it should be easy to do. Since $\sqrt{2} < 1.5$ we see that $3 - 2\sqrt{2}$ is clearly the root closest to the origin.
When analyzing for α we must get 4.901. It seems like this value may actually be difficult to compute in the way presented in these slides. Presumably one should be able to do so as on slide 46 of AC07 by evaluating a squared version of the right side (or potentially solved for S(z) or in this case $S(z)^2$ first, if this is how it is supposed to be solved). However, since the lectures recommend solving for α to be left to computers in many instances, it doesn't seem particularly important for this problem to solve it in this manner, as one can essentially repurpose the radius-of-convergence transfer theorem we learned a while ago to solve for what α must be in this case. Doing so gave me ≈ 4.901 and indeed, on OEIS someone else found an asymptotic approximation for α and it matches my result exactly. The exact quantity is actually $\alpha = \sqrt{12 + 17/\sqrt{2}}$ but even if one uses the method I used, it takes considerable

algebraic manipulation to reach such a nice form.
This α gives our asymptotic approximation to be:

$$F(z)[z^n] \sim \frac{4.901}{2\sqrt{\pi}}(3+2\sqrt{2})^n n^{-3/2}$$

A final note is that, interestingly enough, this asymptotic approximation takes the longest to converge/has the higher error at lower terms out of any of the approximations I've ever evaluated. It's off by so much for lower values, that I thought I had made a mistake, and then when I found that the asymptotics on OEIS matched mine exactly, I thought it possible that both of us had made the same mistake somehow. I had to use Mathematica to expand it out to 50 terms to get an error of 3.3% to be convinced of its correctness. (For instance, term 6 of the asymptotics gives 3687.75 while the true value should be 550. Does singularity analysis tend to give less accurate estimations/ones that converge more slowly than our other tranfer theorems, or was this just happenstance?)