Homework 11: Exercise VI.1

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The symbolic specifications translate to the following system of generating function equations

$$S(z) = 1 + zU(z)S(z) + zD(z)S(z)$$
$$U(z) = z + zU(z)^2$$
$$D(z) = z + zD(z)^2$$

Solving for $S(z)$, we have

$$U(z) = D(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z}$$
$$S(z) = \frac{1}{1 - 4z^2} = \left(1 - 4z^2\right)^{-\frac{1}{2}}$$

Define $f(u) = (1-u)^{-\frac{1}{2}}$, and note that the coefficient of u^n in $f(u)$ can be extracted using the standard function scale theorem:

$$[u^N]\,(1-u)^{-\frac{1}{2}} = \frac{N^{\frac{1}{2}-1}}{\Gamma\left(\frac{1}{2}\right)} \sim \frac{1}{\sqrt{\pi N}}$$

Therefore,

$$[z^{2N}]\left(1 - z^2\right)^{-\frac{1}{2}} \sim \frac{1}{\sqrt{\pi N}}$$

and

$$[z^{2N}]\left(1 - 4z^2\right)^{-\frac{1}{2}} \sim \frac{4^N}{\sqrt{\pi N}},$$

as long as $N > 0$. Note that for a string to have an equal number of zeros and ones, it must have even length. Therefore, the coefficients of z^k for odd k are all zero, and all the relevant information is given by the asymptotics of the coefficients of terms with an even exponent.