Homework 11: Exercise VI.2

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Let $\Omega = \{0, 2, 3\}$ be the set of allowed number of children. We will use the simple variety of trees schema to enumerate rooted plane trees with degrees constrained to Ω.

$$\begin{aligned} \mathcal{F} &= \mathcal{Z} \times \textsc{Seq}_{\Omega}(\mathcal{Z}) \\ F(z) &= z\phi(F(z)) \\ \phi(u) &= 1 + u^2 + u^3. \end{aligned}$$

We will confirm that these trees match the criteria for the transfer theorem for simple varieties of trees. More specifically, we note that $F(z)$ belongs to the smooth inverse-function schema (Definition VII.3. on page 453 of the book), since

- $\phi(z)$ is analytic at 0
- $\phi(0) \neq 0$, $[z^u]\phi(u) \in \{0, 1\} \geqslant 0$ for all n, and $\phi(u) \neq \phi_0 + \phi_1 u$ (since ϕ is a degree 2 polynomial);
- The radius of convergence of ϕ is infinite, and the characteristic equation $\phi(\tau) - \tau\phi'(\tau) = 0$ (or equivalently $1 - u^2 - 2u^3 = 0$), can be verified to have a unique positive solution roughly when $u = 0.657298$.

Therefore, we can apply the transfer theorem for simple varieties of trees (Theorem VII.2, page 453) to derive asymptotics of $[z^n]F(z)$.

Let $\tau = 0.657298$ be the positive root of the characteristic equation and $\rho = \frac{\tau}{\phi(\tau)} = 0.38304$. Then Theorem VII.2 gives

$$\begin{aligned} [z^n]F(z) &\sim \sqrt{\frac{\phi(\tau)}{2\phi''(\tau)}} \frac{\rho^{-n}}{\sqrt{\pi n^3}} \\ &= \boxed{0.2631 \cdot 2.6107^n n^{-3/2}}. \end{aligned}$$

The number of bits required to represent such a tree is $\sim \log\left(0.2631 \cdot 2.6107^n n^{-3/2}\right) = 0.960n - 1.335 - \frac{3\log(n)}{2} = \Theta(n)$.