Homework 11: Exercise VII.1

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As outlined on page 69 of the book, bracketings can be specified as

$$\mathcal{S} = \mathcal{Z} + \textsc{Seq}_{\geqslant 2}(S)$$
$$S(z) = \phi(z, S(z)),$$
$$\phi(z, w) = z + \frac{w^2}{1-w}$$

We will confirm that these functions match the criteria for the transfer theorem for tree-like structures. More specifically, we note that $S(z)$ satisfies the criteria of the smooth implicit-function schema (Definition VII.4. on page 467), since

- $F(z)$ is analytic at zero: $F(0) = \phi(0, F(0)) = 0 + \frac{F(0)^2}{1-F(0)} \Rightarrow F(0) = 0$ or $F(0) = 1/2$, none of which is a singularity.
- $F_0 = 0$ and $F_n \geqslant 0$ for all n, since the coefficients are a counting sequence and there are no empty bracketings.
- $\phi(z, w)$ is analytic in a domain $|z| < R$ and $|w| < S$ for some $R, S > 0$: As long as $w < 1$ and for any z, the function is analytic.
- The coefficients of ϕ satisfy $\phi_{m,n} \geqslant 0, \phi_{0,0} = 0, g_{0,1} \neq 1$:
  This is the case since $\phi(z, w) = z + \sum_{i=2}^{\infty} w^i$.
- $g_{m,n} > 0$ for some m and for some $n \geqslant 2$:
  This holds when $m = 0$ and $n = 2$ for example, since $g_{0,2} = 1 > 0$.
- There exist two numbers r, s, such that $0 < r < R$ and $0 < s < S$, satisfying characteristic system $\phi(r, s) = s, \phi_w(r, s) = 1$ with $r < R$, $s < S$:

Let $0 < r = 3 - 2\sqrt{2}$ and $0 < s = 1 - \frac{1}{\sqrt{2}} < 1$. We can confirm that

$$r = 3 - 2\sqrt{2} < \infty,$$

$$s = 1 - \frac{1}{\sqrt{2}} < 1$$

$$\phi(3 - 2\sqrt{2}, 1 - \frac{1}{\sqrt{2}}) = 3 - 2\sqrt{2} + \frac{\left(1 - \frac{1}{\sqrt{2}}\right)^2}{1 - (1 - \frac{1}{\sqrt{2}})} = 1 - \frac{1}{\sqrt{2}} = s$$

$$\phi_w(3 - 2\sqrt{2}, 1 - \frac{1}{\sqrt{2}}) = \left.\frac{w(2 - w)}{(1 - w)^2}\right|_{w = 1 - \frac{1}{\sqrt{2}}} = 1$$

Therefore, we can apply the transfer theorem for implicit tree-like classes:

$$\alpha = \sqrt{\frac{2r\phi_z(r, s)}{\phi_{ww}(r, s)}} = \sqrt{\frac{2(3 - 2\sqrt{2}) \cdot 1}{\left.\frac{2}{(1-w)^3}\right|_{w = 1 - \frac{1}{\sqrt{2}}}}} = \sqrt{\frac{3\sqrt{2} - 4}{4}}$$

$$[z^N]S(z) \sim \frac{\alpha}{2\sqrt{\pi}} r^{-N} N^{-3/2} = \boxed{0.070 \cdot 5.828^N N^{-3/2}}.$$