Miranda Moore
COS 488/MAT 474
Problem Set 11, Q1

5/5

**AC Web Exercise VI.1** Use the standard function scale to directly derive an asymptotic expression for the number of strings in the following context-free grammar:

$$\mathcal{S} = \epsilon + (\mathcal{U} \times \mathcal{Z} \times S) + (\mathcal{D} \times Z \times S),$$
$$\mathcal{U} = \mathcal{Z} + (\mathcal{U} \times \mathcal{U} \times Z),$$
$$\mathcal{D} = \mathcal{Z} + (\mathcal{D} \times \mathcal{D} \times \mathcal{Z}).$$

*Solution.* These constructions translate into the OGF equations

$$U(z) = z + zU(z)^2, \qquad D(z) = U(z)$$
$$S(z) = 1 + 2zU(z)S(z).$$

Solving the quadratic equation for $U(z)$, we get

$$U(z) = \frac{1 \pm \sqrt{1 - 4z^2}}{2z}$$

and we choose the minus sign in order to make the coefficients positive. Now solving for $S(z)$, we get

$$S(z) = \frac{1}{1 - 2zU(z)} = \frac{1}{\sqrt{1 - 4z^2}}.$$

From the standard function scale, we directly get the coefficient asymptotics

$$[z^{2N}](1 - 4z^2)^{-1/2} \sim \frac{4^N N^{-1/2}}{\Gamma(1/2)} = \frac{4^N}{\sqrt{\pi N}}.$$

Therefore, the number of strings of length $N$ in the context-free grammar $\mathcal{S}$ is

$$[z^N]S(z) = \begin{cases} \sim \frac{4^{N/2}}{\sqrt{\pi N/2}} = \sqrt{\frac{2}{\pi N}}\, 2^N & \text{if } N \text{ is even,} \\ 0 & \text{if } N \text{ is odd.} \end{cases}$$