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COS 488/MAT 474
Problem Set 11, Q2

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AC Web Exercise VI.2 Give an asymptotic expression for the number of rooted ordered trees for which every node has 0, 2, or 3 children. How many bits are necessary to represent such a tree?

Solution. This class of trees has the construction

$$\mathcal{T} = \mathcal{Z} \times SEQ_{0,2,3}(\mathcal{T})$$

which translates to the OGF equation

$$T(z) = z(1 + T(z)^2 + T(z)^3).$$

This is a simple variety of trees, with characteristic function $\phi(u) = 1 + u^2 + u^3$. Let $\lambda \doteq 0.65730$ be the real root of the characteristic equation $\phi(u) = 1 + u^2 + u^3 = u\phi'(u) = 2u^2 + 3u^3$. Let $\alpha = \frac{\phi''(\lambda)}{\phi(\lambda)} = \frac{2+6\lambda}{1+\lambda^2+\lambda^3} \doteq 3.46371$, and let $\beta = \phi'(\lambda) = 2\lambda + 3\lambda^2 \doteq 2.61072$. By the transfer theorem for simple varieties of trees, we have the coefficient asymptotics

Should be beta^n here

$$[z^N]T(z) \sim \frac{1}{\sqrt{2\pi\beta}}\alpha^N N^{-3/2} \doteq (0.21436)(3.46371)^N N^{-3/2}.$$

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Should be 1/sqrt(2 pi alpha) beta^n

This is the approximate number of such trees with N nodes. The number of bits necessary to represent such a tree has to be at least

$$\begin{aligned}\lceil \log_2([z^N]T(z)) \rceil &\dot{\sim} \log_2(.21436) + N\log_2(3.46371) - \frac{3}{2}\log_2(N) \\ &\doteq -2.222 + (1.792)N - \frac{3}{2}\log_2(N).\end{aligned}$$