Miranda Moore
COS 488/MAT 474
Problem Set 11, Q3

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AC Web Exercise VII.1 Use the tree-like schema to develop an asymptotic expression for the number of bracketings with N leaves.

Solution. As shown in the lecture, the class of bracketings has the construction

$$\mathcal{S} = \mathcal{Z} + SEQ_{>1}(\mathcal{S})$$

which translates to the OGF equation

$$S(z) = z + \frac{1}{1 - S(z)} - 1 - S(z).$$

The characteristic function is

$$\Phi(z, w) = z + \frac{1}{1 - w} - 1 - w.$$

Φ is analytic when $|w| < 1 = S$. The characteristic system of equations is

$$\Phi(r, s) = r + \frac{1}{1 - s} - 1 - s = s,$$
$$\Phi_w(r, s) = \frac{1}{(1 - s)^2} - 1 = 1.$$

Solving the second equation for s yields $s = 1 \pm \frac{1}{\sqrt{2}}$, but because we must have $s < S = 1$, we choose the solution $s = 1 - \frac{1}{\sqrt{2}}$. Solving the first equation for r then yields $r = 3 - 2\sqrt{2}$. Now, we can apply the transfer theorem for smooth-implicit-function tree-like classes. We have

$$\Phi_z(r, s) = 1,$$
$$\Phi_{ww}(r, s) = \frac{2}{(1 - s)^3} = 4\sqrt{2},$$
$$\alpha = \sqrt{\frac{2r\Phi_z(r, s)}{\Phi_{ww}(r, s)}} = \sqrt{\frac{r}{2\sqrt{2}}}.$$

Therefore, the coefficient asymptotics are

$$[z^N]S(z) \sim \frac{\alpha}{2\sqrt{\pi}} \left(\frac{1}{r}\right)^N N^{-3/2} = \sqrt{\frac{r}{8\sqrt{2}\pi}} \left(\frac{1}{r}\right)^N N^{-3/2},$$

where $r = 3 - 2\sqrt{2}$.