Miranda Moore
COS 488/MAT 474
Problem Set 11, Q&A
Singularity Analysis

**Question** Let $\mathcal{A}$ be the class of unary-binary trees (ordered, rooted, unlabelled trees where each node has 0, 1, or 2 children) with the following restriction: the left child of each 2-node must be either a leaf or a 1-node. Write a functional equation for the generating function $A(z)$. Show that

$$[z^N]A(z) \sim c\,\varphi^{2N}N^{-3/2},$$

where $c$ is some constant and $\varphi = \frac{1+\sqrt{5}}{2} \doteq 1.618$, the "golden ratio".

(Useful identity: $\varphi^2 = \varphi + 1$.)

*Solution.* A tree in $\mathcal{A}$ is either (1) a node, (2) a node connected to another tree in $\mathcal{A}$, or (3) a node connected to [a node, or a node connected to another tree in $\mathcal{A}$] and a tree in $\mathcal{A}$.

$$\begin{aligned} \mathcal{A} &= \mathcal{Z} + \mathcal{Z} \times \mathcal{A} + \mathcal{Z} \times (\mathcal{Z} + \mathcal{Z} \times \mathcal{A}) \times \mathcal{A} \\ \Rightarrow A(z) &= z + zA(z) + z(z + zA(z))A(z) \\ &= z + zA(z) + z^2 A(z) + z^2 A(z)^2. \end{aligned}$$

$\mathcal{A}$ is an implicit tree-like class with characteristic function $\Phi(z, w) = z + zw + z^2 w + z^2 w^2$. The characteristic system of equations is

$$\begin{aligned} \Phi(z, w) &= z + zw + z^2 w + z^2 w^2 = w \\ \Phi_w(z, w) &= z + z^2 + 2z^2 w = 1 \end{aligned}$$

which has the solution $(z, w) = (1/\varphi^2, \varphi)$ in the positive reals, where $\varphi = \frac{1+\sqrt{5}}{2}$. So $A(z)$ satisfies the axioms for the transfer theorem for implicit tree-like classes, with $r = 1/\varphi^2$ and $s = \varphi$. By this transfer theorem, we automatically get

$$[z^N]A(z) \sim c\, \varphi^{2N} N^{-3/2},$$

where $c$ is a constant (which is somewhat complicated to compute in this case).