COS 488 - Homework 11 - Question & Answer

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1 Question

1. Suppose $\phi(u)$ has non-negative coefficients, is not of the form $\phi_0 + \phi_1 u$, is analytic at 0 with $\phi(0) \neq 0$ and radius of convergence R, and has one positive real root $\lambda < R$ of the equation $\phi(\lambda) = \lambda\phi'(\lambda)$. Then, show that

$$[u^{n-1}]\phi(u)^n \sim \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{\phi(\lambda)}{\phi''(\lambda)}}\phi'(\lambda)^n.$$

2. Use part 1 to show each of the following asymptotic equivalences:

   (a) Stirling's approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

   (b) For fixed integral $k > 1$,

$$\binom{kn}{n} \sim \frac{1}{\sqrt{2\pi n}}\sqrt{1+\frac{1}{k-1}}\left(k\left(1+\frac{1}{k-1}\right)^{k-1}\right)^n.$$

   (c) If Δ_n is the number of ways to partition $\{1, \ldots, n-1\}$ into n parts, each of which has size 0, 1, or 2, then

$$\Delta_n \sim \frac{3^n}{\sqrt{4\pi n/3}}.$$

2 Answer

1. Let $f(z)$ satisfy $f(z) = z\phi(f(z))$. We will calculate $n[z^n]f(z)$ in two different ways.

   Firstly, if $g(u) = \frac{u}{\phi(u)}$, then $g(f(z)) = z$, $g(0) = 0$, and $g'(0) \neq 0$, so by Lagrange inversion, we have

$$n[z^n]f(z) = [u^{n-1}]\,(u)\,g(u)^n = [u^{n-1}]\phi(u)^n.$$

   Secondly, $f(z)$ represents a λ-invertible simple variety of trees, so by the transfer theorem for invertible tree classes, we have

$$n[z^n]f(z) \sim n\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\phi(\lambda)}{\phi''(\lambda)}}\phi'(\lambda)^n n^{-3/2} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{\phi(\lambda)}{\phi''(\lambda)}}\phi'(\lambda)^n.$$

   Equating these two expressions for $n[z^n]f(z)$ gives the desired result.

2. (a) Let $\phi(u) = e^u$, so that $\phi(u)$ satisfies the conditions in part 1 with $\lambda = 1$. Then, since $[u^{n-1}](e^u)^n = \frac{n^{n-1}}{(n-1)!}$, the formula from part 1 gives

$$\frac{n^{n-1}}{(n-1)!} \sim \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{e}{e}}e^n = \frac{e^n}{\sqrt{2\pi n}}.$$

   By rearranging this formula, we get Stirling's approximation.

   (b) Let $\phi(u) = (1+u)^k$, so that $\phi(u)$ satisfies the conditions in part 1 with $\lambda = \frac{1}{k-1}$. Then, we have

$$\binom{kn}{n} \sim (k-1)\binom{kn}{n-1} = (k-1)[u^{n-1}]\phi(u)^n,$$

   so by the formula in part 1, we have

$$\binom{kn}{n} \sim (k-1)\frac{1}{\sqrt{2\pi n}}\sqrt{\frac{\left(1+\frac{1}{k-1}\right)^k}{k(k-1)\left(1+\frac{1}{k-1}\right)^{k-2}}}\left(k\left(1+\frac{1}{k-1}\right)^{k-1}\right)^n$$
$$= \frac{1}{\sqrt{2\pi n}}\sqrt{1+\frac{1}{k-1}}\left(k\left(1+\frac{1}{k-1}\right)^{k-1}\right)^n.$$

   (c) The number of ways to partition $\{1,\ldots,a\}$ into b parts, each of which has size in the set S is

$$[z^a]\left(\sum_{s\in S} z^s\right)^b.$$

   In particular, if $\phi(u) = 1 + u + u^2$, then $\Delta_n = [u^{n-1}]\phi(u)^n$. Since $\phi(u)$ satisfies the conditions in part 1 with $\lambda = 1$, the formula in part 1 gives

$$\Delta_n \sim \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{3}{2}}3^n = \frac{3^n}{\sqrt{4\pi n/3}}.$$