COS 488 Problem Set #11 Test Example Question

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Question

Give an asymptotic approximation for the number of rooted trees of size n that don't contain nodes that have a single child.

Answer

The combinatorial construction is given by $\mathcal{T} = \mathcal{Z} + \mathcal{Z} \times \mathcal{T} \times \mathcal{T} + \mathcal{Z} \times \mathcal{T} \times \mathcal{T} \times \mathcal{T} + \ldots$. As a result, $T(z) = z\left(\frac{1}{1-T(z)} - T(z)\right)$. This is an invertible tree class with $\phi(u) = \frac{1}{1-u} - u$. This clearly has nonnegative coefficients and is analytic and nonzero at zero with positive radius of convergence. We wish to solve the characteristic equation:

$$\begin{aligned}
\phi(\lambda) &= \lambda\phi'(\lambda) \\
\frac{1}{1-\lambda} - \lambda &= \frac{\lambda}{(1-\lambda)^2} - \lambda \\
1 - \lambda &= \lambda \\
\lambda &= 1/2
\end{aligned}$$

Hence, by the transfer theorem for simple varieties of trees, we have

$$\begin{aligned}
[z^n]T(z) &\sim \frac{1}{\sqrt{2\pi\phi''(1/2)/\phi(1/2)}}(\phi'(1/2))^n n^{-3/2} \\
&= \frac{1}{\sqrt{2\pi(16)/(3/2)}} 3^n n^{-3/2} \\
&= \frac{\sqrt{3}}{8\sqrt{\pi}} 3^n n^{-3/2}
\end{aligned}$$