David Luo
Exercise 3.20

5/5

Take 1: $a_0 = 0,\ a_1 = 0,\ a_2 = 1$

First, make the recurrence valid for all values of n:

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} + \delta_{n2}$$

Multiply by z^n and sum up:

$$A(z) = 3zA(z) - 3z^2A(z) + z^3A(z) + z^2$$

$$A(z) = \frac{z^2}{1-3z+3z^2-z^3} = \frac{z^2}{(1-z)^3} = \sum_{N\geq 2} \binom{N}{2} z^N$$

This is a known generating function, and its corresponding series (and the recurrence) is

$$a_n = \frac{n(n-1)}{2}$$

Take 2: $a_0 = 0,\ a_1 = 1,\ a_2 = 1$

First, make the recurrence valid for all values of n:

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} + \delta_{n1} - 2\delta_{n2}$$

Multiply by z^n, sum up, and solve for $A(z)$:

$$A(z) = 3zA(z) - 3z^2A(z) + z^3A(z) + z - 2z^2$$

$$A(z) = \frac{z-2z^2}{1-3z+3z^2-z^3} = \frac{z(1-2z)}{(1-z)^3}$$

Use partial fraction decomposition to find the constants:

$$A(z) = \frac{\alpha}{1-z} + \frac{\beta}{(1-z)^2} + \frac{\gamma}{(1-z)^3} = \frac{z-2z^2}{(1-z)^3}$$

$$\alpha(1-z)^2 + \beta(1-z) + \gamma = z - 2z^2$$

$$-2(1-z)^2 + 3(1-z) - 1 = z - 2z^2,\ \alpha = -2,\ \beta = 3,\ \gamma = -1$$

Add up the known generating functions to solve for the recurrence:

$$\frac{\alpha}{1-z} = \alpha(1 + z + z^2 + ...) = \alpha \sum_{N\geq 0} z^N$$

$$\frac{\beta}{(1-z)^2} = \beta(1 + z + z^2 + ...)(1 + z + z^2 + ...) = \beta(1 + 2z + 3z^2 + ...) = \beta \sum_{N\geq 0} (N+1)z^N$$

$$\frac{\gamma}{(1-z)^3} = \gamma(1 + 2z + 3z^2 + ...)(1 + z + z^2 + ...) = \gamma[1 + (1+2)z + (1+2+3)z^2 + ...] = \gamma \sum_{N\geq 0} \frac{(N+1)(N+2)}{2} z^N$$

$$= \sum_{N\geq 0} (\alpha * 1 + \beta * (N+1) + \gamma\frac{(N+1)(N+2)}{2}) = \sum_{N\geq 0} a_N z^N$$

Thanks for showing your work!

$$a_N = \alpha + \beta(N+1) + \gamma\frac{N^2+3N+2}{2} = -2 + 3(N+1) - (\frac{N^2+3N+2}{2})$$

$$a_N = \frac{1}{2}N(3-N)$$