Analytic Combinatorics Homework 2 Problem 2

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5/5

We have

$$A_N = \frac{N-6}{N} A_{N-1} + 2.$$

Thus, the summation factor is $\frac{N-6}{N} \cdot \frac{N-7}{N-1} \cdot \dots \cdot \frac{1}{7} = \frac{1}{\binom{N}{6}}$. This suggests multiplying both sides by $\binom{N}{6}$. We thus have

$$\binom{N}{6} A_N = \binom{N}{6} \cdot \frac{N-6}{N} A_{N-1} + 2\binom{N}{6} = \binom{N-1}{6} A_{N-1} + 2\binom{N}{6}.$$

Let $B_N = \binom{N}{6} A_N$. We have $B_N = B_{N-1} + 2\binom{N}{6}$ and $B_5 = \binom{5}{6} A_5 = 0$, so

$$B_N = 2\left(\binom{N}{6} + \binom{N-1}{6} + \dots + \binom{6}{6}\right) = 2\binom{N+1}{7},$$

where the last step is by the hockey stick identity. If $\binom{N}{6} \neq 0$, i.e. $N \geq 6$, we have

$$A_N = \frac{B_N}{\binom{N}{6}} = \frac{2(N+1)N \dots (N-5) \cdot 6!}{N(N-1) \dots (N-5) \cdot 7!} = \frac{2(N+1)}{7}.$$

For $N < 6$ we may simply compute the values A_N, and we get $A_1 = 1$ (given); $A_2 = 0$; $A_3 = 2$; $A_4 = 1$; and $A_5 = \frac{9}{5}$. Thus, we have

$$\boxed{A_1 = 1, A_2 = 0, A_3 = 2, A_4 = 1, A_5 = \frac{9}{5}, A_N = \frac{2(N+1)}{7} \quad \forall N \geq 6}.$$

How does this relate to the average number of 2-nodes after N steps?