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Analytic Combinatorics Homework 2 Problem 3

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Let $A(z)$ be the generating function for the sequence. The recurrence relation gives us $A(z) = 3zA(z) - 3z^2A(z) + z^3A(z) + z^2$, where z^2 is necessary because $a_2 - 3a_1 + 3a_0 = 1$. We thus have

$$(z-1)^3A(z) = -z^2$$
$$A(z) = \frac{z^2}{(1-z)^3}.$$

Now, we have $\frac{1}{1-z} = 1 + z + z^2 + \dots$. Taking the second derivative of both sides, we have

$$\frac{2}{(1-z)^3} = 2 \cdot 1 + 3 \cdot 2z + 4 \cdot 3z^2 + \cdots = \sum_{k=0}^{\infty}(k+1)(k+2)z^k.$$

Therefore, we have

$$A(z) = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} z^{k+2} = \sum_{k=2}^{\infty} \frac{k(k-1)}{2} z^k = \sum_{k=0}^{\infty} \frac{k(k-1)}{2} z^k$$

(where the last step comes from $\frac{0(-1)}{2} = \frac{1 \cdot 0}{2} = 0$). Therefore, we have

$$\boxed{a_n = \frac{n(n-1)}{2}}.$$

If instead we let $a_1 = 1$, the equation for our generating function changes so that the extra terms are $z - 2z^2$ (we get z because $a_1 - 3a_0 = 1$ and $-2z^2$ because $a_2 - 3a_1 + 3a_0 = -2$). Thus we have

$$A(z) = \frac{z - 2z^2}{(1-z)^3},$$

so

$$A(z) = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} z^{k+1} - 2\sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} z^{k+2} = \sum_{k=1}^{\infty} \frac{k(k+1)}{2} z^k - \sum_{k=2}^{\infty} k(k-1)z^k.$$

Again we may make the sums start with 0 because $\frac{0 \cdot 1}{2} = 1 \cdot 0 = 0(-1) = 2$, so we have

$$A(z) = \sum_{k=0}^{\infty} \left(\frac{k(k+1)}{2} - k(k-1) \right) z^k = \sum_{k=0}^{\infty} \frac{k(3-k)}{2} z^k.$$

Therefore, we now have

$$\boxed{a_n = \frac{n(3-n)}{2}}.$$