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Analytic Combinatorics Homework 2 Problem 4

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On the one hand, we have

$$\frac{d}{d\alpha}(1-z)^{-\alpha} = \frac{d}{d\alpha}e^{-\alpha\ln(1-z)} = -\ln(1-z)e^{-\alpha\ln(1-z)} = (1-z)^{-\alpha}\ln\frac{1}{1-z}.$$

On the other hand we have

$$\begin{aligned}(1-z)^{-\alpha} &= 1^{-\alpha} - z\binom{-\alpha}{1} + z^2\binom{-\alpha}{2} - z^3\binom{-\alpha}{3} + \dots \\ &= 1 - z(-\alpha) + z^2\cdot\frac{-\alpha(-\alpha-1)}{2!} - z^3\cdot\frac{-\alpha(-\alpha-1)(-\alpha-2)}{3!} + \dots \\ &= 1 + \alpha z + \frac{\alpha(\alpha+1)}{2!}z^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}z^3 + \dots .\end{aligned}$$

Differentiating, we have

$$\frac{d}{d\alpha}(1-z)^{-\alpha} = z + \frac{\alpha(\alpha+1)}{2!}\left(\frac{1}{\alpha}+\frac{1}{\alpha+1}\right)z^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}\left(\frac{1}{\alpha}+\frac{1}{\alpha+1}+\frac{1}{\alpha+2}\right)z^3 + \dots .$$

Setting our two expressions for $\frac{d}{d\alpha}(1-z)^{\alpha}$ equal and plugging in $\alpha = \frac{1}{2}$, we find that

$$\frac{1}{\sqrt{1-z}}\ln\frac{1}{1-z} = \sum_{n=1}^{\infty}\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\dots\cdot\frac{2n-1}{2}}{n!}\left(2+\frac{2}{3}+\frac{2}{5}+\dots+\frac{2}{2n-1}\right)z^n.$$

Thus we have

$$[z^n]\frac{1}{\sqrt{1-z}}\ln\frac{1}{1-z} = \frac{(2n-1)!!\sum_{k=1}^{n}\frac{2}{2k-1}}{2^n n!} = \frac{(2n-1)!!\sum_{k=1}^{n}\frac{1}{2k-1}}{2^{n-1}n!}.$$

(Note that this also extends to the case of $n = 0$, where the sum in the numerator is a sum of zero elements and is thus zero, and the denominator is nonzero, giving 0, the correct coefficient in the generating function based on our calculations above.)

Your final answer is acceptable but a little sloppy--note that (2n-1)!!/(2^(n-1)*n!) is (2nCn)/(2^(2n-1))