COS 488
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PS2 - Q3

First we want to make this equation valid for all n. For $n = 0$, all the indices are negative and so $a_0 = 0$ as needed. For $n = 1$, we have that $a_1 = 3a_0 = 0$, again as needed. For $n = 2$, we have that $a_2 = 3a_1 - 3a_0 = 0$, when we need $a_2 = 1$, so we add a Kronecker delta term for $n = 2$, giving the recurrence:

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} + \delta_{n2} \qquad \text{for } n \geq 0$$

Now we multiply both sides by z^n and sum over all values of n to get;

$$A(z) = 3zA(z) - 3z^2A(z) + z^3A(z) + z^2$$

Rearranging the terms gives:

$$A(z)(1 - 3z + 3z^3 - z^3) = z^2$$

$$A(z) = \frac{z^2}{1 - 3z + 3z^3 - z^3} = \frac{z^2}{(1-z)^3}$$

Note that this last expression is precisely the OGF for the sequence $0, 0, 1, 3, 6, 10, ...$ where $a_n = \binom{n}{2}$ that we see in the lecture slides, and that we obtain by differentiating twice the OGF

$$\frac{1}{1-z} = \sum_{n \geq 0} z^n$$

.

Hence, we have that

$$A(z) = \frac{z^2}{(1-z)^3} = \sum_{n \geq 2} \binom{n}{2} z^n$$

$$a_n = [z^n]A(z) = \binom{n}{2}$$

And since the binomial coefficient is 0 for $n = 0, n = 1$, this is the general solution for the sequence:

$$a_n = \binom{n}{2} \qquad \text{for } n \geq 0$$

Now, let's redo the problem under the assumption that $a_1 = 1$. As before, the initial recurrence satisfies $a_0 = 0$ as needed. Now for $n = 1$, the recurrence gives $a_1 = 0$ while we need $a_1 = 1$, so we need to add a Kronecker delta term for $n = 1$. For $n = 2$, the recurrence now gives $a_2 = 3$ while we need $a_2 = 1$, so we need to subtract 2 times a Kronecker delta term for $n = 2$. This gives us the general recurrence:

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} + \delta_{n1} - 2\delta_{n2} \qquad \text{for } n \geq 0$$

Multiplying both sides by z^n and summing over n, we get:

$$A(z) = 3zA(z) - 3z^2A(z) + z^3A(z) + z - 2z^2$$

Rearranging terms, we get:

$$A(z)(1 - 3z + 3z^3 - z^3) = z - 2z^2$$

$$A(z) = \frac{z - 2z^2}{1 - 3z + 3z^3 - z^3} = \frac{z - 2z^2}{(1-z)^3} = \frac{z}{(1-z)^2} - \frac{z^2}{(1-z)^3}$$

Now we note that the first term here is the OGF

$$\frac{z}{(1-z)^2} = \sum_{n \geq 1} nz^n$$

And the second term is the OGF

$$\frac{z^2}{(1-z)^3} = \sum_{n \geq 2} \binom{n}{2} z^n$$

And hence:

$$[z^n]A(z) = [z^n]\frac{z}{(1-z)^2} - [z^n]\sum_{n \geq 2} \binom{n}{2} z^n = n - \binom{n}{2}$$

Again, the values match up for $a_0 = 0 - \binom{0}{2} = 0, a_1 = 1 - \binom{1}{2} = 1, a_2 = 2 - \binom{2}{2} = 1$, and so this is our general solution:

$$a_n = n - \binom{n}{2} \qquad \text{for } n \geq 0$$