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COS 488
Eric Wu - ezwu@
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PS2 - Q4

Consider an expression of the form $(1-z)^{-\alpha}$ where α is a positive real number. We can expand this expression using its Taylor series as:

$$(1-z)^{-\alpha} = 1 + \alpha z + \frac{\alpha(\alpha+1)}{2!}z^2 + ...$$

Now note that taking the derivative of this equation with respect to alpha gives

$$\frac{d}{d\alpha}\left((1-z)^{-\alpha}\right) = \frac{d}{d\alpha}\left(1 + \alpha z + \frac{\alpha(\alpha+1)}{2!}z^2 + ...\right)$$

On the LHS, the derivative evaluates to:

$$\frac{d}{d\alpha}\left((1-z)^{-\alpha}\right) = \frac{d}{d\alpha}\left(e^{-\alpha\ln(1-z)}\right) = e^{-\alpha\ln(1-z)}(-\ln(1-z)) = \frac{1}{(1-z)^{\alpha}}\ln\frac{1}{1-z}$$

Note that for $\alpha = 0.5$, we have exactly the OGF that we wish to know the sequence for.

To evaluate the RHS derivative, we examine write the RHS as:

$$\frac{d}{d\alpha}\left(1 + \alpha z + \frac{\alpha(\alpha+1)}{2!}z^2 + ...\right) = 0 + \sum_{N\geq 1}\left(\frac{z^N}{N!}\frac{d}{d\alpha}\left(\prod_{0\leq j\leq N-1}(\alpha+j)\right)\right)$$

Now note that:

$$\frac{d}{d\alpha}\left(\prod_{0\leq j\leq N}(\alpha+j)\right) = \frac{d}{d\alpha}\left((\alpha+N)\prod_{0\leq j\leq N-1}(\alpha+j)\right)$$

$$= (\alpha+N)\frac{d}{d\alpha}\left(\prod_{0\leq j\leq N-1}(\alpha+j)\right) + \prod_{0\leq j\leq N-1}(\alpha+j)$$

The derivation of b_n is pretty roundabout here--why not just immediately iterate the product rule? (f_1...f_n)' = (f_1...f_n)*sum(f_i'/f_i)

This is a recurrence relationship, where we can define:

$$b_N = \frac{d}{d\alpha}\left(\prod_{0\leq j\leq N}(\alpha+j)\right) \qquad \text{for } N > 0$$

with $b_0 = 1$ the base condition.

The recurrence then takes the form:

$$b_N = (\alpha+N)b_{N-1} + \prod_{0\leq j\leq N-1}(\alpha+j) \qquad \text{for } N > 0$$

Dividing the equation by $\prod_{0\le j\le N}(\alpha+j)$, we get:

$$\frac{b_N}{\prod_{0\le j\le N}(\alpha+j)} = \frac{b_{N-1}}{\prod_{0\le j\le N-1}(\alpha+j)} + \frac{1}{\alpha+N} \qquad \text{for } N>0$$

which then telescopes to:

$$\frac{b_N}{\prod_{0\le j\le N}(\alpha+j)} = \frac{b_0}{\alpha} + \sum_{1\le j\le N}\frac{1}{\alpha+j} \qquad \text{for } N>0$$

Plugging in $b_0=1$ and multiplying, we have:

$$b_N = \left(\sum_{0\le j\le N}\frac{1}{\alpha+j}\right)\left(\prod_{0\le j\le N}(\alpha+j)\right) \qquad \text{for } N>0$$

At $N=0$, note that this expression turns into $\frac{\alpha}{\alpha}=1$ and hence the closed form formula is

$$b_N = \left(\sum_{0\le j\le N}\frac{1}{\alpha+j}\right)\left(\prod_{0\le j\le N}(\alpha+j)\right) \qquad \text{for } N\ge 0$$

Now, we can rewrite our RHS in terms of these b_N:

$$\sum_{N\ge 1}\left(\frac{z^N}{N!}\frac{d}{d\alpha}\left(\prod_{0\le j\le N-1}(\alpha+j)\right)\right) = \sum_{N\ge 1}\left(\frac{b_{N-1}}{N!}z^N\right)$$

And now, using the equality

$$\frac{1}{(1-z)^\alpha}\ln\frac{1}{1-z} = \sum_{N\ge 1}\left(\frac{b_{N-1}}{N!}z^N\right)$$

We have that

$$[z^N]\frac{1}{(1-z)^\alpha}\ln\frac{1}{1-z} = \frac{b_{N-1}}{N!} \qquad \text{for } N>0$$

and

$$[z^0]\frac{1}{(1-z)^\alpha}\ln\frac{1}{1-z} = 0$$

Plugging in $\alpha=0.5$, we get

$$[z^N]\frac{1}{(1-z)^{0.5}}\ln\frac{1}{1-z} = \frac{b_{N-1}}{N!} = \frac{1}{N!}\left(\sum_{0\le j\le N-1}\frac{1}{0.5+j}\right)\left(\prod_{0\le j\le N-1}(0.5+j)\right) \qquad \text{for } N>0$$

$$[z^0]\frac{1}{(1-z)^{0.5}}\ln\frac{1}{1-z} = 0$$