5/5, slightly late submissions are fine

A2_Q4

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Begin with the generating function for the catalan numbers:

$$zT(z) = \frac{1}{2}(1 - \sqrt{1-4z})$$

take the derivative:

$$= \frac{1}{\sqrt{1-4z}}$$

(Thus making the terms of the sequence of catalan numbers go from $\frac{1}{N+1}2Nz^N$ to $\frac{1}{N+1}2Nz^{N+1}$ to $2Nz^{N+1}$.)
Now if we multiply each term of the sequence by $1/4^n$ we get the generating function: $\frac{1}{1-z}$, (with terms $2Nz^{N+1}/4^N$ which we can convolve with the generating function for the harmonic sequence (not series), $\ln \frac{1}{1-z}$, giving the desired generating function, $\frac{1}{\sqrt{1-z}} \ln \frac{1}{1-z}$, with term coefficients:

$$[z_n] = \sum_{k=0}^{N-1} \binom{2k}{k} \frac{1}{4^k(N-k)}.$$

(Worked with Maryam B.) This was my original solution, which I submitted before the deadline at the time of the other problems. I had originally tried to solve the problem using the hint, and got an ugly sum/product because I had made a small algebra error. Then this way seemed to make more sense after watching the catalan-numbers generating-function lectures (since we just have the two equations and convolve them). Unfortunately, right after 12am, Matt Tyler and I found a much much nicer solution (this is perhaps the nicest solution I've ever seen to a problem on a pset). So here it is, I know this is a bit past the deadline because I had to type it up and decide what to do, since the lateness policy isn't posted nor made clear etc., so if you don't accept small-deltas-past submissions, please just grade the above solution. If it's fine, the below one is much nicer. Thank you.
Using Taylor's theorem we see that:

$$(1-z)^{-a} = 1 + az + \frac{a(a+1)z^2}{2} + \frac{a(a+1)(a+2)z^3}{3!} + \ldots$$

and thus if we take the derivative with respect to a, we get:

$$\frac{d}{da}(a-z)^{-a} = \frac{d}{da} e^{a \ln \frac{1}{1-z}}$$

$$= \frac{1}{(1-z)^a} \ln \frac{1}{1-z}$$

The desired generating function. If we apply the derivative to the terms above, we see that for a coefficient of z^n of the final generating function, we have:

$$\frac{d}{da}\left(\frac{a(a+1)(a+2)...(a+n-1)}{n!}\right) = \frac{\prod_{i=0}^{n-1}(a+i) \cdot \sum_{i=0}^{n-1}(\frac{1}{a+i})}{n!}$$

and now we evaluate this at a = 1/2, as per the question:

$$\frac{\prod_{i=0}^{n-1}(1/2+i) \cdot \sum_{i=0}^{n-1}(\frac{1}{1/2+i})}{n!} =$$

$$\frac{\frac{1}{2^n}\prod_{i=0}^{n-1}(1+2i) \cdot 2\sum_{i=0}^{n-1}(\frac{1}{1+2i})}{n!} = \frac{(2n-1)!!(H_{2n} - H_n/2)}{2^{n-1}n!} = \frac{\frac{(2n)!}{2^n(n!)}(2H_{2n} - H_n)}{2^n n!} =$$

$$\frac{(2n)!(2H_{2n} - H_n)}{n!n!4^n} = \binom{2n}{n}\frac{2H_{2n} - H_n}{4^n}$$

Just a beautiful solution. And it highlights an interesting identity, setting the first and second solutions equal we see that:

$$\sum_{k=0}^{N-1}\binom{2k}{k}\frac{1}{4^k(N-k)} = \binom{2n}{n}\frac{2H_{2n} - H_n}{4^n}$$

a result that looks like it should be easy to find, but is very difficult to find without solving this question the two different ways.