Homework 2: Exercise 3.20

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a) We will solve the recurrence with initial conditions $a_0 = 0, a_1 = 0, a_2 = 1$ and

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} \quad \text{for } n > 2$$

Hypothesizing that a linear recurrence of this form has a solution asymptotically growing as z^n, we have the following equation:

$$\begin{aligned} z^3 &= 3z^2 - 3z + 1 \\ (z-1)^3 &= 1 \\ z &= 1. \end{aligned}$$

In the future I'd prefer if you were a bit more explicit as to why the solution must be in this form.

Since the generating function has a single root with multiplicity 3, the closed form expression has the following form: $a_n = (A + Bn + Cn^2)1^n$. The initial conditions can be used to solve for the constants A, B, and C:

$$\begin{aligned} a_0 &= A = 0 \\ a_1 &= A + B + C = 0 \\ a_2 &= A + 2B + 4C = 1 \\ \implies A &= 0, B = -\frac{1}{2}, C = \frac{1}{2}. \end{aligned}$$

The closed form is thus given by

$$a_n = \frac{1}{2}n^2 - \frac{1}{2}n = \binom{n}{2}.$$

Finally, let us confirm inductively that this expression holds for $n > 2$:

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} = 3\binom{n-1}{2} - 3\binom{n-2}{2} + \binom{n-3}{2} = \binom{n}{2}.$$

b) We will now solve the recurrence with a different set of initial conditions, $a_0 = 0, a_1 = 1, a_2 = 1$. Solving for the constants A, B, and C again, we have

$$\begin{aligned} a_0 &= A = 0 \\ a_1 &= A + B + C = 1 \\ a_2 &= A + 2B + 4C = 1 \\ \implies A &= 0, B = \frac{3}{2}, C = -\frac{1}{2}. \end{aligned}$$

The recurrence can thus be specified by the following closed form:

$$a_n = -\frac{n(n-3)}{2}.$$

For $n > 2$, we can confirm via induction that

$$\begin{aligned} a_n &= 3a_{n-1} - 3a_{n-2} + a_{n-3} \\ &= -3\frac{(n-1)(n-4)}{2} + 3\frac{(n-2)(n-5)}{2} - \frac{(n-3)(n-6)}{2} \\ &= -\frac{n(n-3)}{2}. \end{aligned}$$