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Homework 2: Exercise 3.28

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First, starting from the generating function $C(z)$ for Catalan numbers, we have

$$\begin{aligned} C(z) &= \frac{1-\sqrt{1-4z}}{2z} \\ C(\frac{z}{4}) &= 2 \cdot \frac{1-\sqrt{1-z}}{z} \\ zC(\frac{z}{4}) &= 2(1-\sqrt{1-z}) \\ \frac{\partial}{\partial z} zC(\frac{z}{4}) &= \frac{1}{\sqrt{1-z}}. \end{aligned}$$

Therefore,

$$\begin{aligned} [z^n]\frac{1}{\sqrt{1-z}} &= [z^n]\frac{\partial}{\partial z} zC(\frac{z}{4}) \\ &= (n+1)[z^{n+1}]zC(\frac{z}{4}) && \text{[OGF differentiation]} \\ &= (n+1)[z^n]C(\frac{z}{4}) && \text{[left shift]} \\ &= \frac{n+1}{4^n}[z^n]C(z) && \text{[scaling]} \\ &= \frac{n+1}{4^n}\frac{\binom{2n}{n}}{n+1} && \text{[def'n of } C(z)\text{]} \\ &= \frac{\binom{2n}{n}}{4^n} && \text{[algebra].} \end{aligned}$$

We are interested in the convolution of $1/\sqrt{1-z}$ function with the generating function for

Harmonic numbers:

$$
\begin{aligned}
[z^n]\frac{1}{\sqrt{1-z}}\ln\frac{1}{1-z} &= \sum_{k=0}^{n}\left([z^k]\frac{1}{\sqrt{1-z}}\cdot\right)\left([z^{n-k}]\ln\frac{1}{1-z}\right) && \text{[def'n of convolution]}\\
&= \sum_{k=0}^{n}\frac{1}{n-k}\cdot[z^k]\frac{1}{\sqrt{1-z}} && \text{[GF for Harmonic numbers]}\\
&= \sum_{k=0}^{n}\frac{1}{n-k}\cdot[z^k]\left(\frac{\partial}{\partial z}zC(\frac{z}{4})\right) && \text{[derivation above]}\\
&= \sum_{k=0}^{n}\frac{1}{n-k}\cdot(k+1)[z^{k+1}]\left(zC(\frac{z}{4})\right) && \text{[OGF differentiation]}\\
&= \sum_{k=0}^{n}\frac{1}{n-k}\cdot(k+1)[z^k]\left(C(\frac{z}{4})\right) && \text{[left shift]}\\
&= \sum_{k=0}^{n}\frac{1}{n-k}\cdot(k+1)\frac{1}{4^k}[z^k]\left(C(z)\right) && \text{[scaling]}\\
&= \sum_{k=0}^{n}\frac{1}{n-k}\cdot(k+1)\frac{1}{4^k}\frac{1}{k+1}\binom{2k}{k} && \text{[def'n of C(z)]}\\
&= \sum_{k=0}^{n}\frac{1}{4^k(n-k)}\binom{2k}{k} && \text{[algebra].}
\end{aligned}
$$

This is technically correct (and I appreciate the clear, detailed writeup), but the final form isn't very simplified. Following the solution implied by the hint should lead to the nicer form (2nCn)(2H_{2n} - H_n)/4^n, where H_n = sum_{i =< n} 1/i.