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COS 488/MAT 474
Problem Set 2, Q2

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AofA Exercise 2.17 Solve the recurrence

$$A_N = A_{N-1} - \frac{2A_{N-1}}{N} + 2\left(1 - \frac{2A_{N-1}}{N}\right)$$

for $N > 1$ with $A_1 = 1$.

Solution. For all $N \geq 1$, we have

$$\begin{aligned} A_N &= \left(1 - \frac{2}{N} - \frac{4}{N}\right) A_{N-1} + 2 \\ &= \frac{N-6}{N} A_{N-1} + 2. \end{aligned}$$

For small values of N, i.e. when $2 \leq N \leq 6$, we can calculate the values of A_N directly from this recurrence:

$$A_2 = 0,\ A_3 = 2,\ A_4 = 1,\ A_5 = \frac{9}{5},\ A_6 = 2.$$

When $N > 6$, we solve the recurrence as follows. Multiply both sides by the summation factor $\frac{N!}{(N-6)!}$ to obtain:

$$\begin{aligned} \frac{N!}{(N-6)!} A_N &= \frac{(N-1)!}{(N-7)!} A_{N-1} + \frac{2(N)!}{(N-6)!} \\ &= \frac{6!}{0!} A_6 + 2 \sum_{k=7}^{N} \frac{k!}{(k-6)!} && \text{(telescope)} \\ &= 2 \sum_{k=6}^{N} \frac{k!}{(k-6)!} \\ &= 2 \cdot 6! \sum_{k=6}^{N} \binom{k}{6} \\ &= 2 \cdot 6! \cdot \binom{N+1}{7} \\ &= \frac{2}{7} \cdot \frac{(N+1)!}{(N-6)!} \\ \implies A_N &= \frac{2(N+1)}{7} \qquad \text{for all } N > 6. \end{aligned}$$

How does this relate to the average number of 2-nodes after N steps?