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COS 488/MAT 474
Problem Set 2, Q3

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**AofA Exercise 3.20** Solve the recurrence

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$$

for $n > 2$ with two sets of initial conditions:

(a) $a_0 = 0,\ a_1 = 0,\ a_2 = 1;$

(b) $a_0 = 0,\ a_1 = 1,\ a_2 = 1.$

*Solution.* First, we observe that by repeated differentiation of both sides, we obtain the Taylor series expansion for $\frac{1}{(1-z)^3}$:

$$\begin{aligned}
(1-z)^{-1} &= \sum_{n=0}^{\infty} z^n \\
(1-z)^{-2} &= \sum_{n=0}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty} (n+1)z^n \\
2(1-z)^{-3} &= \sum_{n=0}^{\infty} n(n+1)z^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2)z^n.
\end{aligned}$$

Thus we have $\frac{1}{(1-z)^3} = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)z^n$.

(a)

$$\begin{aligned}
a_n &= 3a_{n-1} - 3a_{n-2} + \delta_{n2} \\
A(z) &= 3zA(z) - 3z^2A(z) + z^3A(z) + z^2 \\
(-z^3 + 3z^2 - 3z + 1)A(z) &= z^2 \\
A(z) &= \frac{z^2}{(1-z)^3} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)z^{n+2} \\
\implies a_n &= \frac{1}{2}(n-1)n.
\end{aligned}$$

(b)

$$\begin{aligned}
a_n &= 3a_{n-1} - 3a_{n-2} + \delta_{n1} - 2\delta_{n2} \\
A(z) &= 3zA(z) - 3z^2A(z) + z^3A(z) + z - 2z^2 \\
(-z^3 + 3z^2 - 3z + 1)A(z) &= z - 2z^2 \\
A(z) &= \frac{z - 2z^2}{(1-z)^3} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)z^{n+1} - \sum_{n=0}^{\infty}(n+1)(n+2)z^{n+2} \\
\implies a_n &= \frac{1}{2}n(n+1) - (n-1)n \\
&= \frac{1}{2}(3-n)n.
\end{aligned}$$