Miranda Moore
COS 488/MAT 474
Problem Set 2, Q4

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AofA Exercise 3.28 Find an expression for

$$[z^n]\frac{1}{\sqrt{1-z}}\ln\frac{1}{1-z}.$$

Solution. First, we use Taylor's theorem to expand $f(z) = (1-z)^{-\alpha}$ for an arbitrary constant α. We calculate the nth derivatives of $f(z)$ as follows:

$$\begin{aligned}
f(z) &= (1-z)^{-\alpha}\\
f'(z) &= \alpha(1-z)^{-\alpha-1}\\
f''(z) &= \alpha(\alpha+1)(1-z)^{-\alpha-2}\\
&\vdots\\
f^n(z) &= \left(\prod_{k=0}^{n-1}(\alpha+k)\right)(1-z)^{-\alpha-n} \qquad \Longrightarrow \qquad f^n(0) = \left(\prod_{k=0}^{n-1}(\alpha+k)\right) \text{ for all } n \geq 0.
\end{aligned}$$

Therefore, the Taylor series expansion of $(1-z)^{-\alpha}$ is

$$\left(\frac{1}{1-z}\right)^{\alpha} = \sum_{n=0}^{\infty}\left(\prod_{k=0}^{n-1}(\alpha+k)\right)\frac{z^n}{n!}.$$

Next, we differentiate both sides with respect to α (using the product rule on the right side):

$$\left(\frac{1}{1-z}\right)^{\alpha}\ln\left(\frac{1}{1-z}\right) = \frac{d}{d\alpha}\left(\frac{1}{1-z}\right)^{\alpha} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n-1}\frac{1}{\alpha+k}\prod_{j=0}^{n-1}(\alpha+j)\right)\frac{z^n}{n!}.$$

Setting $\alpha = \frac{1}{2}$, we obtain

$$\begin{aligned}
\frac{1}{\sqrt{1-z}}\ln\left(\frac{1}{1-z}\right) &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n-1}\frac{2}{1+2k}\prod_{j=0}^{n-1}\frac{1+2j}{2}\right)\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n-1}\frac{1}{1+2k}\prod_{j=0}^{n-1}(1+2j)\right)\frac{z^n}{2^{n-1}n!}.
\end{aligned}$$

Therefore, the coefficient of z^n is

$$[z^n]\frac{1}{\sqrt{1-z}}\ln\frac{1}{1-z} = \frac{1}{2^{n-1}n!}\left(\sum_{k=0}^{n-1}\frac{1}{1+2k}\right)\left(\prod_{j=0}^{n-1}(1+2j)\right).$$