COS 488 - Homework 2 - Question 2

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Let $A_1 = 1$, and for all $N > 1$, let

$$A_N = A_{N-1} - \frac{2A_{N-1}}{N} + 2\left(1 - \frac{2A_{N-1}}{N}\right) = \frac{(N-6)A_{N-1}}{N} + 2,$$

so that the first few values of A_N are given in the table below:

N	1	2	3	4	5	6
A_N	1	0	2	1	9/5	2

Now, assume that $N > 6$. When we divide this recurrence by the summation factor

$$\frac{N-6}{N}\frac{N-7}{N-1}\cdots\frac{1}{7} = \frac{1}{N(N-1)(N-2)(N-3)(N-4)(N-5)}$$

and also divide by the constant 6!, we get

$$\binom{N}{6}A_N = \binom{N-1}{6}A_{N-1} + 2\binom{N}{6},$$

which telecopes as

$$\binom{N}{6}A_N = \binom{N-1}{6}A_{N-1} + 2\binom{N}{6} = \binom{N-2}{6}A_{N-2} + 2\binom{N-1}{6} + 2\binom{N}{6} = A_6 + 2\sum_{k=7}^{N}\binom{N}{6} = 2\binom{N+1}{7}$$

by the Hockey-stick identity. Therefore, for all $N > 6$ (and indeed for all $N \geq 6$),

$$A_N = \frac{2\binom{N+1}{7}}{\binom{N}{6}} = \frac{2(N+1)}{7}.$$

In conclusion, we have:

$$A_N = \begin{cases} 1 \text{ if } N = 1 \\ 0 \text{ if } N = 2 \\ 2 \text{ if } N = 3 \\ 1 \text{ if } N = 4 \\ 9/5 \text{ if } N = 5 \\ \frac{2(N+1)}{7} \text{ if } N \geq 6 \end{cases}.$$

How does this relate to the average number of 2-nodes after N steps?